Computing Nash Equilibria in Multidimensional Congestion Games

Extended Abstract

Mohammad T. Irfan
Department of Computer Science
Bowdoin College
Brunswick, Maine, United States
mirfan@bowdoin.edu

Hau Chan
School of Computing
University of Nebraska-Lincoln
Lincoln, Nebraska, United States
hchan3@unl.edu

Jared Soundy
School of Computing
University of Nebraska-Lincoln
Lincoln, Nebraska, United States
jared.soundy@huskers.unl.edu

ABSTRACT

We study pure-strategy Nash equilibrium (PSNE) computation in k-dimensional congestion games (k-DCGs) where the weights or demands of the players are k-dimensional vectors. We first show that deciding the existence of a PSNE in a k-DCG is NP-complete even for games when players have binary and unit demand vectors. We then focus on computing PSNE for k-DCGs and their variants with general, linear, and exponential cost functions. For general cost functions (potentially non-monotonic), we provide the first configuration-space framework to find a PSNE if one exists. For linear and exponential cost functions, we provide potential function-based algorithms to find a PSNE. These algorithms run in polynomial time under certain assumptions. We also study structured demands and cost functions, giving polynomial-time algorithms to compute PSNE for several cases. For general cost functions, we give a constructive proof of existence for an (α, β)-PSNE (for certain α and β), where α and β are multiplicative and additive approximation factors, respectively.

KEYWORDS

Non-cooperative Game Theory, Congestion Games, Nash Equilibrium, Algorithms, Computational Complexity

ACM Reference Format:
Mohammad T. Irfan, Hau Chan, and Jared Soundy. 2024. Computing Nash Equilibria in Multidimensional Congestion Games: Extended Abstract. In *Proc. of the 23rd International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2024), Auckland, New Zealand, May 6 – 10, 2024*, IFAAMAS, 3 pages.

MULTIDIMENSIONAL CONGESTION GAMES

In congestion games [17], there are a number of resources (e.g., edges in a road network). Each player has a set of strategies, where each strategy is a subset of resources (e.g., source-destination paths in a road network). A strategy profile consists of a strategy for each player. The cost of a resource (e.g., edge) is a function of the number of players using that resource. Under a given strategy profile, the cost to a player is the sum of the costs of the resources used by the player in their own strategy. A strategy profile is called a pure-strategy Nash equilibrium (PSNE) if no player can unilaterally decrease their cost by choosing a different strategy.

Congestion games can be broadly divided into three main classes: unweighted [1, 4, 15, 17], weighted [3, 5–9, 12–14, 16], and multidimensional [10, 11]. Unweighted congestion games are the classical ones described in the previous paragraph. In a weighted congestion game, each player has a weight or demand. Multidimensional congestion games extend weighted congestion games by making the weight a multidimensional vector. Very recently, Klimm and Schütz [11] have characterized the existence of PSNE in k-dimensional congestion games (k-DCGs). Their characterization leads to the following computational questions, which we address here: *How can we compute a PSNE (if it exists) in multidimensional congestion games and their variants? How hard is this computation?*

Formally, a k-dimensional congestion game (k-DCG) consists of a set $N = \{1, \dots, n\}$ of n players and a set $R = \{1, \dots, m\}$ of m resources. Each player i has two elements: (1) a strategy set $S_i \subseteq 2^R \setminus \{\emptyset\}$ and (2) a k-dimensional demand vector $\mathbf{d}_i = (d_{i_1}, ..., d_{i_k}) \in \mathbb{R}^k$. Each resource r has a cost function $c_r : \mathbb{R}^k \to \mathbb{R}$. We use p to denote the maximum number of strategies of any player.

Given a strategy profile $\mathbf{s} = (s_1, ..., s_n) \in S = S_1 \times ... \times S_n$, let $\mathbf{x}_r(\mathbf{s}) = \sum_{i \in N: r \in s_i} \mathbf{d}_i$ be the aggregated k-dimensional demand vector of the players who select resource r under $\mathbf{s}$. Given a strategy profile $\mathbf{s}$, the cost function of player i is defined to be $\pi_i(\mathbf{s}) = \pi_i(s_i, \mathbf{s}_{-i}) = \sum_{r \in s_i} c_r(\mathbf{x}_r(\mathbf{s}))$. A strategy profile $\mathbf{s}^*$ is a pure-strategy Nash equilibrium (PSNE) in a k-DCG if and only if for each player i and any $s'_i \in S_i$, $\pi_i(\mathbf{s}^*) \le \pi_i(s'_i, \mathbf{s}^*_{-i})$.

We are interested in computing pure-strategy Nash equilibria of k-DCGs and their variants listed below.

- k-DCGs with binary demand vectors $\mathbf{d}_i \in \{0, 1\}^k \; \forall i$.
- k-class congestion games (k-CCGs), where each demand vector has one positive element, the rest being zeros.
- k-DCGs with player types, where players of the same type are characterized by the same demand vector.

COMPUTATIONAL COMPLEXITY

We show that deciding the existence of a PSNE in special variants of k-DCGs is NP-complete. The following results are interesting in the context of known hardness results, such as NP-completeness of deciding PSNE in weighted congestion games (i.e., $k = 1$) [3].

Theorem 1. *Deciding the existence of a PSNE in a k-DCG is NP-complete even when the demand vector $\mathbf{d}_i$ of each player $i \in N$ is a binary vector and k is sublinear in the number of players. That is, $\mathbf{d}_i \in \{0, 1\}^k$ for all i and $k = O(\log n)$.*

Proc. of the 23rd International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2024), N. Alechina, V. Dignum, M. Dastani, J.S. Sichman (eds.), May 6 – 10, 2024, Auckland, New Zealand. © 2024 International Foundation for Autonomous Agents and Multiagent Systems (www.ifaamas.org).

Theorem 2. *Deciding the existence of a PSNE in a k-DCG (or a k-CCG) is NP-complete even when the demand vector* $\mathbf{d}_i$ *of each player* $i \in N$ *is a binary unit vector and k is linear of the number of players. That is,* $\mathbf{d}_i \in \{\mathbf{x} \in \{0,1\}^k; \sum_{j=1}^{k} x_j = 1\}$ *for all i and* $k = O(n)$.

GENERAL COST FUNCTIONS

For k-DCGs and their variants with general cost functions (potentially non-monotonic), we present algorithms that are polynomial in n (the number of players), p (the maximum number of strategies for any player), and a maximum weight term when k (number of dimensions) and m (number of resources) are bounded. This is useful for congestion games in which there is a constant number of dimensions and strategies with many players.

Our algorithmic framework explores the configuration space (i.e., the space of aggregated demand vectors) for computing a PSNE. It works by exploring the possible aggregated demand vectors of all players and verifying (non-trivially) whether an aggregated demand vector can lead to a PSNE. For variants of k-DCGs, such as k-DCGs with player types, k-DCGs with binary demands, and k-CCGs, we exploit their structures to derive the following results.

In the following results, we define $w_{\max} = \max_{j \in [k]} w_j$, where $w_j = \sum_{i \in N} d_{ij}$ for each $j = 1, ..., k$.

Theorem 3. *For any k-DCG, there is an algorithm to determine the existence of a PSNE in* $O((w_{\max})^{km}(nkp^2m^2 + nkmp(w_{\max})^{km}))$*, which is polynomial in n, p, and* $w_{\max}$*, when m and k are constants.*

In the following result on k-DCGs with binary demand, we use $\check{n} = \max_{j \in [k]} \sum_{i \in N} d_{ij}$ in place of $w_{\max}$ to represent the maximum number of players having any particular demand vector bit on. This result improves the running time given by Theorem 3.

Theorem 4. *For k-DCGs with binary demand vectors, there is an* $O(\check{n}^{km}(nkp^2m^2 + \min\{nkmp\check{n}^{km}, n^{km+1}p\}))$*-time algorithm to compute a PSNE or decide that there exists none. The algorithm is polynomial in n and p when m and k are constants.*

We use a partitioning technique to derive the following result for k-CCGs, improving the running time given by Theorem 3.

Theorem 5. *For k-CCGs, there is an* $O\left((w_{\max})^{km}(np^2m^2 + nkpm(w_{\max})^m)\right)$*-time algorithm to compute a PSNE or decide that there exists none. The algorithm is polynomial in n, p, and* $w_{\max}$ *when m and k are constants.*

We get the following corollary from Theorems 4 and 5.

Corollary 1. *For k-CCGs with binary demand vectors, there is an* $O((\check{n})^{km}(np^2m^2 + nkpm(\check{n})^m))$*-time algorithm to compute a PSNE or decide that there exists none. The algorithm is polynomial in n and p when m and k are constants.*

We have the following result for k-DCGs with player types.

Theorem 6. *Given a k-DCG with* τ *types of players and at most* $\check{n}$ *players of any type, there is an* $O((\check{n})^{\tau m}(np^2m^2 + n\tau pm(\check{n})^m) + \tau nk)$ *time algorithm to compute a PSNE or decide that there exists none. The algorithm is polynomial in n and p for bounded m and* τ.

Computing Approximate PSNE

A strategy profile $\mathbf{s}$ is an (α, β)-PSNE in a k-DCG for some $\alpha \geq 1$ and $\beta \geq 0$ if and only if for each player i and any $s_i' \in S_i$, $\pi_i(\mathbf{s}) \leq \alpha \pi_i(s_i', \mathbf{s}_{-i}) + \beta$. The following result generalizes the result in [2] by removing the monotonicity assumption while retaining the non-negative cost assumption. Here, $\Delta_{\max}$ is the maximum non-negative marginal decrease of any player due to deviation.

Theorem 7. *Every k-DCG has an* (α, β)*-PSNE for* $\alpha = n$ *and* $\beta = (n-1)m\Delta_{\max}$*. Furthermore, it can be computed using an iterative algorithm that is guaranteed to converge.*

STRUCTURED COSTS AND DEMANDS

The following results utilize structural information regarding the cost functions (e.g., how the costs of resources compare with each other), demand vectors (e.g., whether the players can be ordered by their demand vectors), and strategies.

Theorem 8. *For a k-DCG with ordered demand vectors, nondecreasing cost functions, and singleton-resource strategies, a PSNE can be computed in* $O(n \log n + nmk)$ *time.*

Theorem 9. *For a k-DCG with ordered demand vectors, nondecreasing cost functions, and a shared set of strategies of size p, a PSNE can be computed in* $O(n \log n + npmk)$ *time.*

Theorem 10. *For a k-DCG with nondecreasing and structured cost functions where there are constants* $\alpha_j \geq 1$ *such that* $c_{j-1}(\mathbf{x}) = \alpha_j c_j(\mathbf{x})$ *for any resource* $j > 1$ *and aggregate demand vector* $\mathbf{x}$*, and singleton-resource strategies, a PSNE can be computed in* $O(n \log n + nmk)$ *time.*

LINEAR AND EXPONENTIAL COST

For linear and exponential cost, for which existence of PSNE in k-DCGs is guaranteed [11], we give potential function-based iterative best-response algorithms by bounding weighted potential functions. First, let the linear cost function of any resource r under a strategy profile $\mathbf{s}$ be $c_r(\mathbf{x}_r(\mathbf{s})) \equiv a_r \sum_{j \in [k]} z_j \mathbf{x}_{r,j}(\mathbf{s}) + b_r = a_r[\mathbf{z} \cdot \mathbf{x}_r(\mathbf{s})] + b_r$, where $a_r, b_r \geq 0$ for all r and the k-dimensional vector $\mathbf{z} \geq 0$. We have the following results on k-DCGs and their variants

Theorem 11. *The best-response algorithm runs in polynomial time if* $\max_r a_r, \max_r b_r$*, and* $\frac{\max_i [\mathbf{z} \cdot \mathbf{d}_i]^2}{\min_i [\mathbf{z} \cdot \mathbf{d}_i]}$ *are polynomial in n.*

Theorem 12. *When the demands are binary vectors, the algorithm runs in polynomial time if the following cost function parameters are polynomial in n:* $\max_r a_r$, $\max_r b_r$*, and* $\max_j z_j$.

Theorem 13. *For k-CCG, the algorithm runs in polynomial time if* $\max_r a_r, \max_r b_r, \frac{\max_j z_j^2}{\min_j z_j}$*, and* $\frac{\max_i d_{i,l(i)}^2}{\min_i d_{i,l(i)}}$ *are polynomial in n, where* $l(i) \in [k]$ *denotes the index of the non-zero element in* $\mathbf{d}_i$.

Finally, for exponential cost functions $c_r(\mathbf{x}_r(\mathbf{s})) = a_r \exp(\mathbf{z} \cdot \mathbf{x}_r(\mathbf{s})) + b_r$, the following result characterizes the running time of the iterative best-response algorithm explicitly.

Theorem 14. *The best-response algorithm runs in polynomial time for k-DCGs with exponential cost functions if* $\max_r a_r$ *and* $\max_r b_r$ *are polynomial in n and* $[\mathbf{z} \cdot \mathbf{d}_N]$ *is* $O(\log n)$.

ACKNOWLEDGMENTS

MTI is grateful for support from NSF Award IIS-1910203. HC thanks NIH for Award P20GM130461 and NSF for Award IIS-2302999.

REFERENCES

[1] Heiner Ackermann, Heiko Röglin, and Berthold Vöcking. 2008. On the Impact of Combinatorial Structure on Congestion Games. *Journal of the ACM (JACM)* 55, 6 (2008), 1–22.

[2] George Christodoulou, Martin Gairing, Yiannis Giannakopoulos, Diogo Poças, and Clara Waldmann. 2023. Existence and Complexity of Approximate Equilibria in Weighted Congestion Games. *Mathematics of Operations Research* 48, 1 (2023), 583–602.

[3] Juliane Dunkel and Andreas S Schulz. 2008. On the Complexity of Pure-Strategy Nash Equilibria in Congestion and Local-Effect Games. *Mathematics of Operations Research* 33, 4 (2008), 851–868.

[4] Alex Fabrikant, Christos Papadimitriou, and Kunal Talwar. 2004. The Complexity of Pure Nash Equilibria. In *Proceedings of the 36th ACM Symposium on Theory of Computing - STOC 2004*. ACM Press, Chicago, IL, USA, 604. https://doi.org/10.1145/1007352.1007445

[5] Dimitris Fotakis, Spyros Kontogiannis, Elias Koutsoupias, Marios Mavronicolas, and Paul Spirakis. 2002. The Structure and Complexity of Nash Equilibria for a Selfish Routing Game. In *International Colloquium on Automata, Languages, and Programming*. Springer, Spain, 123–134.

[6] Dimitris Fotakis, Spyros Kontogiannis, and Paul Spirakis. 2005. Selfish Unsplittable Flows. *Theoretical Computer Science* 348, 2 (Dec. 2005), 226–239. https://doi.org/10.1016/j.tcs.2005.09.024

[7] Martin Gairing, Thomas Lücking, Marios Mavronicolas, and Burkhard Monien. 2004. Computing Nash Equilibria for Scheduling on Restricted Parallel Links. In *Proceedings of the 36th ACM Symposium on Theory of Computing - STOC 2004*. ACM Press, Chicago, IL, USA, 613–622.

[8] Tobias Harks and Max Klimm. 2012. On the Existence of Pure Nash Equilibria in Weighted Congestion Games. *Mathematics of Operations Research* 37, 3 (Aug. 2012), 419–436. https://doi.org/10.1287/moor.1120.0543 Publisher: INFORMS.

[9] Tobias Harks, Max Klimm, and Rolf H Möhring. 2011. Characterizing the Existence of Potential Functions in Weighted Congestion Games. *Theory of Computing Systems* 49, 1 (2011), 46–70.

[10] Max Klimm and Andreas Schütz. 2014. Congestion Games with Higher Demand Dimensions. In *Web and Internet Economics (Lecture Notes in Computer Science)*, Tie-Yan Liu, Qi Qi, and Yinyu Ye (Eds.). Springer International Publishing, Cham, 453–459. https://doi.org/10.1007/978-3-319-13129-0_39

[11] Max Klimm and Andreas Schütz. 2022. Equilibria in Multiclass and Multidimensional Atomic Congestion Games. *Mathematics of Operations Research* 47, 4 (2022), 2743–2764.

[12] L. Libman and A. Orda. 1997. Atomic Resource Sharing in Noncooperative Networks. In *Proceedings of INFOCOM '97*, Vol. 3. IEEE, Japan, 1006–1013 vol.3. https://doi.org/10.1109/INFCOM.1997.631115 ISSN: 0743-166X.

[13] Marios Mavronicolas, Igal Milchtaich, Burkhard Monien, and Karsten Tiemann. 2007. Congestion Games with Player-Specific Constants. In *Mathematical Foundations of Computer Science 2007*, Luděk Kučera and Antonín Kučera (Eds.). Springer, Berlin, Heidelberg, 633–644. https://doi.org/10.1007/978-3-540-74456-6_56

[14] Igal Milchtaich. 1996. Congestion Games with Player-Specific Payoff Functions. *Games and Economic Behavior* 13, 1 (March 1996), 111–124. https://doi.org/10.1006/game.1996.0027

[15] Dov Monderer and Lloyd S Shapley. 1996. Potential Games. *Games and economic behavior* 14, 1 (1996), 124–143.

[16] Panagiota N. Panagopoulou and Paul G. Spirakis. 2007. Algorithms for Pure Nash Equilibria in Weighted Congestion Games. *ACM Journal of Experimental Algorithmics* 11 (Feb. 2007), 2–7. https://doi.org/10.1145/1187436.1216584

[17] Robert W. Rosenthal. 1973. A Class of Games Possessing Pure-Strategy Nash Equilibria. *International Journal of Game Theory* 2, 1 (Dec. 1973), 65–67. https://doi.org/10.1007/BF01737559