Zeroes of Generalized Zeta Functions

Arav Agarwal, Class of 2023

Number theory is the branch of mathematics that deals with the properties and relationships of numbers, particularly integers. It has a rich history that dates back to ancient times, with famous mathematicians such as Euclid, Diophantus, Euler, and Fermat making significant contributions. The field's influence is not relegated to antiquity, and it continues to maintain its status as a principal branch of mathematics. Host to a variety of open problems and long-standing conjectures waiting to be resolved, number theory has been one of the fundamental guides and motivators for the very manner in which modern mathematics has developed. German mathematician Carl Friedrich Gauss (1777 - 1855) said, "Mathematics is the queen of the sciences – and number theory is the queen of mathematics."

With this context, we can now begin an explication of the research Prof. Tanabe and I conducted this summer. One of the most important functions in number theory is the Riemann-zeta function. By a function, we simply mean a mathematical object which takes an input, and gives some output. The Riemann-zeta function is defined as the following infinite sum:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots \quad ,$$

where s is a number and acts as the input value. This mathematical function is the primary inspiration behind our research. The Riemann-zeta function has many important generalizations, some examples of which include Dirichlet L-functions and Dedekind zeta functions. Just like the Riemann-zeta function, understanding the location of their zeros is a lively area of mathematical research, and is one of Prof. Tanabe's areas of specialization. Specifically, our research project involved understanding the density of the zeros of these functions around the critical line. Research is largely an incremental game: understanding something about the zeros of generalized zeta functions would be incremental progress towards conjectures such as the generalized Riemann hypothesis, and similarly has profound implications on the distribution of prime numbers, and finds broad applicability in physics and cryptography.

I am happy to report that we have made substantial progress towards our goal. In particular, we are well on our way to obtaining an "explicit formula" for the "averaged 1-level density" of "$\mathfrak{f}$-ray class functions". Roughly speaking, that translates to obtaining a useful and illustrative formula relating to a certain class of interesting functions. We are in a position to write a strong thesis in terms of my Honors project, and also expect to soon have a publishable paper with novel results.

Over the summer, Prof. Tanabe and I would have weekly meetings, where we would work through difficult proofs together, and also set the agendas for our individual workload over the next week. In the time between meetings, we would communicate using a shared project file and email, and share our progress with each other. There were certainly moments, as there always are in mathematics research, where we encountered very difficult hurdles which we were not sure how to solve. Thankfully, they were always eventually overcome with perseverance and lots of time. Overall, the summer was very productive in terms of research progress, and I am very grateful to the college for their generosity, and providing me with this opportunity.

Faculty mentor: Prof. Naomi Tanabe.

Funded by the Kibbe Science Fellowship.