A Computation of $L(2k+1, \chi_4)$

Nawapan Wattanawanichkul, Class of 2021

The Riemann-zeta function is one of the functions that mathematicians have considered tremendously due to its applications. This function is closely related to the distribution of prime numbers, which is essential to the modern world information security. Further, the usage of this function is widespread in several fields other than mathematics, such as physics, probability theory, and applied statistics. Despite a vast contribution of the function, its definition is a simple-looking infinite series of the form,

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots, \tag{1}$$

where s is any complex number whose real part is greater than 1. For any infinite series, one of the frequently asked questions is if there is a formula for its exact values. In fact, in the case of the Riemann-zeta function, it is well-known that its values can be expressed as

$$\zeta(2k) := \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{2k-1} \pi^{2k}}{(2k)!} B_{2k}, \tag{2}$$

where k is any positive integer and $\{B_{2k}\}$ is a certain sequence of integers, called Bernoulli numbers.On the other hand, there is no known formula for odd integers $2k+1$ that we can directly evaluate their values.

Indeed, formula (2) has been proven in many ways, yet in 2015, the authors presented in [1] a new simple way to show it. Their methods merely rely on a telescopic technique and some basic trigonometric identities. For clarification, the telescopic technique is a way to rewrite every term in a series in such a way that most of the terms get canceled, leaving us an easy-to-compute summation. The authors did not only use this technique to recreate the proof of (2) but also to obtain an integral representation of the function evaluated at positive odd integers.

The straightforward yet creative computation in [1] inspired us to extend their results to other functions. As a consequent, we shifted our focus from the Riemann-zeta function to one of its generalized forms called Dirichlet L-function. The Dirichlet L-functions are defined similarly to (1) but multiplying each term by a value of a certain function called a Dirichlet character χ, i.e.,

$$L(k, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^k}. \tag{3}$$

Since there are infinitely many such characters χ, we focused on a specific character, namely the primitive character modulo 4. In contrast to (2), now we only know the formula for the L-function evaluated at odd integers, which is

$$L(2k+1, \chi_4) = (-1)^{k+1} \left(\frac{\pi}{2}\right)^{2k+1} \frac{B_{2k+1,\chi_4}}{(2k+1)!}. \tag{4}$$

Although our approach is to follow what is used in [1], the appearance of character values $\chi_4(n)$ forced us to modify most of the details. In truth, instead of studying the properties of generalized Bernoulli numbers directly, we consider another object called Euler numbers, E_k, which is very closely related to B_{k,χ_4}. The object played a significant role in obtaining the formula given in (4). Another challenge we faced was to find an appropriate auxiliary function that yields the formula (4). We had to refine our function creatively and precisely. At last, we were able to obtain the formula by evaluating the function,

$$I^*(k,m) := \int_0^{1/2} \left(E_{2k}(t) - \frac{E_{2k}}{2^{2k}} \sin(\pi t)\right) \sin((2m+1)\pi t)\, dt$$

in two different ways.

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References:

[1] Ó. Ciaurri, L. M. Navas, F. J. Ruiz, and J. L. Varona, *A simple computation of* $\zeta(2k)$. The American Mathematical Monthly 122, No. 5, 2015.