Classifying Flow-kick Equilibria:
Reactivity and Transient behavior in the Variational Equation

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In recent years, the scientific community has become increasingly alarmed by current and projected reports of rapid climate change. Global warming, rising sea levels, melting ice caps, and increased frequency of extreme weather events threaten ecosystems which we are reliant on both for resources and for the sustained biodiversity of the planet.[1] In light of these growing concerns, scientists are asking more questions about the resilience of these ecosystems to environmental disturbances. Among these questions are those which address how sustainable management techniques might be used to maintain the desired stable states of ecosystems.

To simulate the ways that management techniques might influence ecosystems subject to environmental disturbance, we turn to mathematical modeling. In particular, we use flow-kick dynamical systems to model periodically managed ecosystems and quantify their resilience to environmental change and management strategies. Flow-kick systems consist of repeated periods of undisturbed, natural growth (flow), each followed by a discrete intervention (kick) (see Figure 1).[2,3,4]

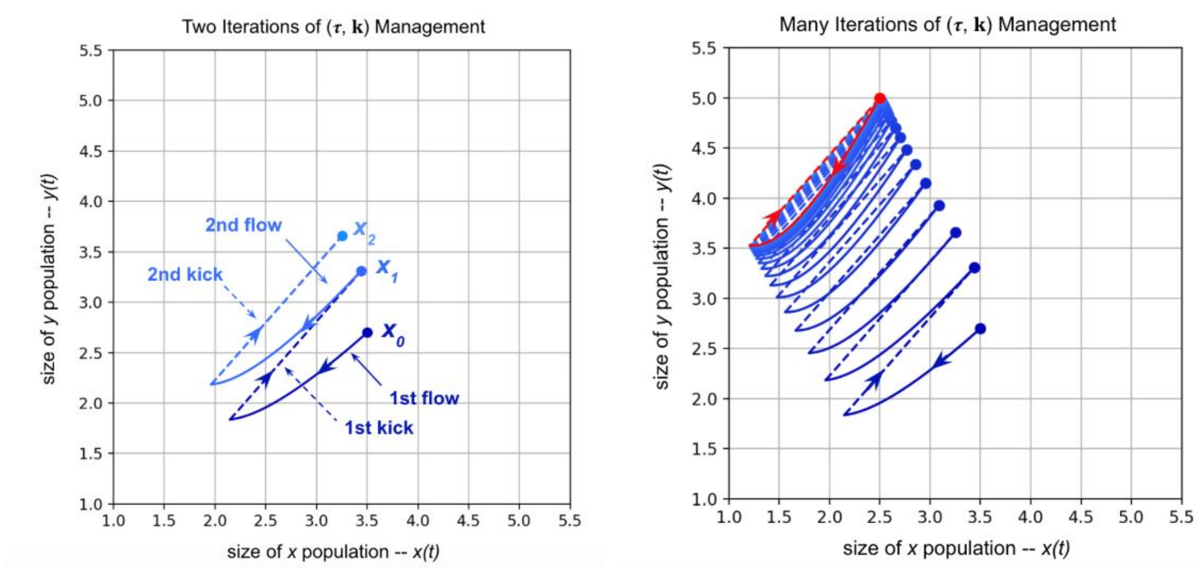

Figure 1. Visualization of a flow-kick dynamical system. The image on the left shows two iterations of flow and kick. The image on the right shows how the behavior of this flow-kick system settles after many iterations.

Flow-kick systems provide a mathematical framework for answering questions such as: *How will this management strategy affect the long term stable state of the system? How resilient would the system be to disturbances in a chosen management strategy?* We aim to answer these questions by developing analytic techniques for classifying the stability of the equilibria of flow-kick systems. This begins with the study of the variational equation which linearly approximates the non-linear behavior of the flow-kick map. The variational equation is constructed by stringing together the instantaneous linearizations of the underlying flow about each point on the equilibrium trajectory (see Figure 2).[5]

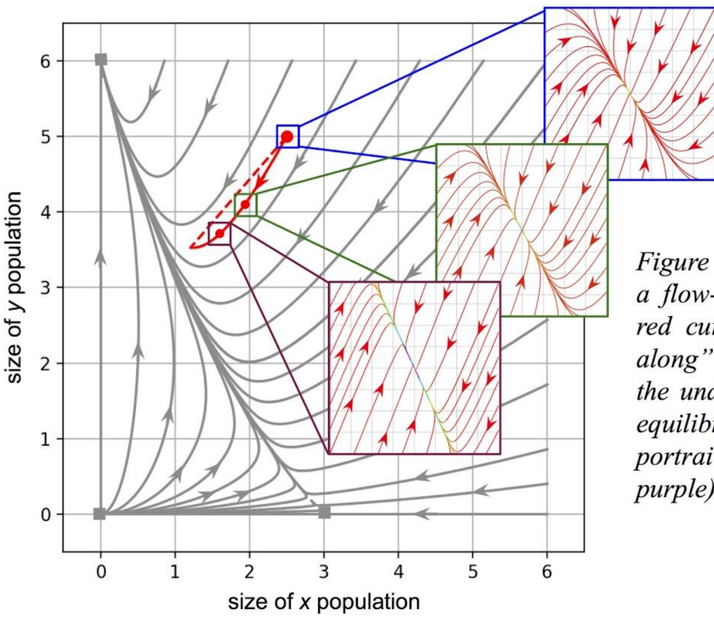

Figure 2. The variational equation about a flow-kick equilibrium trajectory (solid red curve) is constructed by "stringing along" the instantaneous linearization of the underlying flow at each point on the equilibrium trajectory (the linear phase portraits outlined in blue, green, and purple).

Using the intuition of "stringing along" instantaneous linear approximations, we conjecture that if the instantaneous linear approximation at every point on the equilibrium trajectory has the same eigenvalue classification, then the asymptotic stability of the time-τ variational map will also match that eigenvalue classification. If the conjecture is true, then in cases when the instantaneous linearizations have the same asymptotic classification, the accumulation of their transient contributions must match that asymptotic behavior. Since proving or disproving this conjecture depends on understanding the difference between transient and asymptotic behavior, we ask the following questions: *To what degree can transient behavior differ from asymptotic behavior? Under what conditions can this transient behavior accumulate asymptotically?*

Pursuing the answers to these questions has involved developing a radial and tangential approach to analysing two-dimensional autonomous linear systems. From this framework, pictured in Figure 3, we can establish conditions under which the conjecture must hold: if the radial velocity is always positive or always negative, the transient contribution must have the same characterization as the eigenvalues. This framework also provides the tools to understand situations in which a non-autonomous linear system could possibly drive the accumulation of transient behavior that differs from the asymptotic.

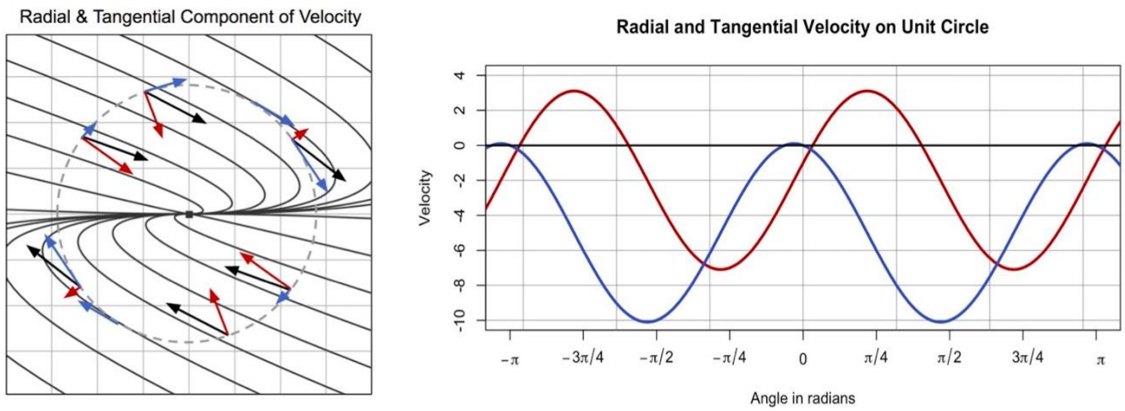

Figure 3. Pictured on the left, the direction of the velocity at each point on the unit circle represents the direction of the velocity for all radial lines. This velocity can be broken into radial and tangential components. Pictured on the right, a plot of the radial and tangential components as a function of angle. The red curve corresponds the the radial velocity and the blue corresponds the the tangential velocity. We can see that they are sinusoidal with period π and are related by a phase shift of $\pi/2$.

From this framework, we have constructed an example of a non-autonomous linear system for which the classification of the time-τ map differs from the asymptotic eigenvalue classification at each instantaneous linear system. This example is not yet sufficient to disprove the conjecture since it may not be a variational equation. Through formalizing this example, we uncovered many structural properties of the radial and tangential framework. We also built intuition which is currently driving numerical experiments that seek to understand which behavior can be observed in the variational equation of Lotka-Voltera competition models and which behavior does not arise.

This work culminated in an honors thesis for the department of mathematics.[6]

Faculty Mentor: Prof. Mary Lou Zeeman

Funded by the NSF

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