Exploration of Kochen-Specker Colorability

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Abstract:
Kochen-Specker Colorability is at the forefront of new work in mathematics. It is based on coloring orthogonal triples (three vectors all at right angles to each other).[3] The question I explored is which three dimensional sets of vectors are Kochen-Specker Colorable, meaning, in which sets can all vectors be colored black or white so that every orthogonal triple has one and only one white vector. Using the work of Michael Pun and Nikhil Dasgupta, past student researchers with Professor Reyes who were working with vectors mod 12, I had previously proven that *Proj*$(\mathbb{M}_3(\mathbb{Z}[\frac{1}{2*3}]))$ was Uncolorable mod 6 and thus discovered a flaw in an earlier attempt to prove *Proj*$(\mathbb{M}_3(\mathbb{Z}[\frac{1}{2*3}]))$ is Kochen-Specker Colorable.[4] In addition, I have proved that *Proj*$(\mathbb{M}_3(\mathbb{Z}[\frac{1}{2*3}]))$ is Kochen-Specker Uncolorable mod 12. Moving forward, I wrote and used a coloring code in Mathematica to prove that *Proj*$(\mathbb{M}_3(\mathbb{Z}[\frac{1}{2*3*7*11}]))$ is Kochen-Specker Uncolorable. My research suggested the possible benefits of moving away from five as an especially significant prime and the value of exploring combinations of primes in the denominator of the adjoined fraction. In addition, the code I developed will serve as a useful tool for future research.

Project Objectives:
My objectives for this summer were to more fully explore Quantum Physics to place my understanding of Kochen-Specker Colorability in a larger context. In addition, I wanted to develop proofs of the Kochen-Specker Colorability of different sets of vectors, particularly *Proj*$(\mathbb{M}_3(\mathbb{Z}[\frac{1}{2*3}]))$. I did not strictly prove this; however, my research not only made progress in the field of Kochen-Specker Colorability, it also paved the way for future discoveries.

Methodology:
I began the summer by gaining a fuller understanding of Quantum Physics so that I could fully understand the importance of Kochen-Specker Colorability. In the following weeks I first attempted to color *Proj*$(\mathbb{M}_3(\mathbb{Z}[\frac{1}{2*3}]))$ by hand. However, I quickly realized that, when not working modulo an integer, the calculations rapidly became too difficult and time consuming for me to complete by hand. I then began working on a program in Mathematica that would allow me to create a set of vectors using the restrictions I entered and color that set by making a few decisions about basic vectors. From there, I was able to prove that *Proj*$(\mathbb{M}_3(\mathbb{Z}[\frac{1}{2*3*7*11}]))$ is Kochen-Specker Uncolorable.

Results Obtained:
Kochen-Specker Colorability rests on the process of coloring orthogonal triples. Geometry often involves coloring, designating each element of a set a specific color. It is a convenient way of categorizing items in a set. A key question is which three dimensional sets of vectors are Kochen-Specker Colorable.

The existence of different three-dimensional sets is counterintuitive for most readers. In this case, it simply means only including certain integer vectors (or vectors with whole number components) in a set. For instance, *Proj*$(\mathbb{M}_3(\mathbb{Z}[\frac{1}{2*3*5}]))$ is a significant set of vectors for Kochen-Specker Colorability.[1] *Proj*$(\mathbb{M}_3(\mathbb{Z}[\frac{1}{2*3*5}]))$ simply means all the integer vectors, (a,b,c), where $a^2+b^2+c^2$ is only divisible by 2, 3, and 5. It has been established that S(2, 3, 5) (meaning the set of vectors where the norm squared is only divisible by 2, 3, or 5 or some combination of them) is Kochen-Specker Uncolorable (meaning there is no way to color the vectors so that every orthogonal triple had one and only one white vector); however, it remained unclear whether five had to be in the denominator of the adjoined fraction for the set to be Kochen-Specker Uncolorable.[1]

Entering my summer research, I had worked on the question of Kochen-Specker Colorability for a year as an independent study. Having already proven that *Proj*$(\mathbb{M}_3(\mathbb{Z}[\frac{1}{2*3}]))$ was Uncolorable mod 6, I had discovered a flaw in an earlier attempt to prove *Proj*$(\mathbb{M}_3(\mathbb{Z}[\frac{1}{2*3}]))$ is Kochen-Specker Colorable. In addition, after beginning my summer work, I developed a proof that *Proj*$(\mathbb{M}_3(\mathbb{Z}[\frac{1}{2*3}]))$ is uncolorable mod 12. However, despite suggesting that modular arithmetic may not be the most efficient method for exploring Kochen-Specker Colorability (as it seemed to be yielding very little insight), my work did not answer the questions surrounding the relationship five has to Kochen-Specker Colorability.

To answer this query, I developed a code that allowed me to quickly color large sets by designating a few vectors white and black. This required me to become familiar with Mathematica, a coding method that was utterly new to me. However, once I mastered the basics, I was able to write a flexible code that colored large sets and could be altered to provide information about how the coloring developed. The discoveries I made with this code are significant; however, the code itself may prove the most useful result for future Kochen-Specker Colorability research. It allows one to explore extremely large sets not accessible when working by hand.

Using this code, I discovered that *Proj*$(\mathbb{M}_3(\mathbb{Z}[\frac{1}{2*3*7*11}]))$ is Kochen-Specker Uncolorable. This confirmed the existence of Kochen-Specker Uncolorable sets where five is not in the denominator of the adjoined fraction, providing vital insight and suggesting that five may be less significant than previously thought. In addition, this new information suggests a way forward for this research.

Significance and Interpretation of Results:

My work indicates the possible benefit of moving away from the idea of five as a vector that must be in the denominator of the adjoined fraction for Kochen-Specker UnColorability. Furthermore, my proof that *Proj*$(\mathbb{M}_3(\mathbb{Z}[\frac{1}{2*3*7*11}]))$ is Kochen-Specker Uncolorable suggests there may be something significant about combinations of primes and may become part of a future publication from Professor Reyes. Thus, my work is a valuable jumping off point for future researchers in the field of Kochen-Specker Colorability.

Kochen-Specker Colorability is an extremely interesting field. First emerging from Quantum Physics, it has been linked to the world of Quantum Computing, which may enable computers to run insurmountably long codes in a fraction of the time.[2] This would be a crucial step forward in the realm of cryptography, allowing for the decryption of currently uncrackable ciphers.

References:

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