Practice Problems: Mathematical Foundations
(Induction, Summations, Recurrences)

1. Order the following expressions by their asymptotic growth and justify your answer.

\[ 2^n, n!, (\log n)!, n^3, e^n, 2^{\log_2 n}, n \log n, 2^{2^n}, n^{\log \log n}. \]

2. Show that the following function are in growth order:

\[ \log \log n, \log n, \sqrt{n}, n, n \log \log n, n \log n, n \log^2 n, n^2, n^3, 2^n. \]

3. Prove by induction that \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \).

4. Solve the recurrence:

\[ T(n) = \begin{cases} 
1 & \text{if } n = 1 \\
T(n - 1) + n(n - 1) & \text{if } n \geq 2
\end{cases} \]

*Hint:* use \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \).

5. Show that \( \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \) for any integer \( n \geq 1 \).

6. Solve the recurrence:

\[ T(n) = \begin{cases} 
1 & \text{if } n = 1 \\
T(n - 1) + n(n - 1)(n+1) & \text{if } n \geq 2
\end{cases} \]

*Hint:* use \( \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \).

7. Give asymptotic upper and lower bounds for \( T(n) \) in each of the following recurrences. Assume \( T(n) \) is constant for sufficiently small \( n \). Make your bounds as tight as possible, and justify your answers.

(a) \( T(n) = T(n - 1) + \lg n. \)

(b) \( T(n) = \sqrt{n} T(\sqrt{n}) + n. \)

(c) \( T(n) = 4 T(n/2) + n^2 \sqrt{n}. \)
8. Show using induction (the substitution method) that the recurrence

\[ T(n) = \begin{cases} 
1 & \text{if } n = 1 \\
2T\left(\frac{n}{2}\right) + b \log n & \text{if } n \geq 2 
\end{cases} \]

where \( b \) is a positive constant has solution \( T(n) = O(n) \).

*Hint:* Show that there exist positive constants \( a \) and \( c \) such that \( T(n) \leq an - b \log n - c \).

9. Show using induction (the substitution method) that the recurrence

\[ T(n) = \begin{cases} 
\sqrt{n} \cdot T(\sqrt{n}) + an & \text{if } n > 2 \\
1 & \text{otherwise}
\end{cases} \]

(where \( a \) is a positive constant) has solution \( T(n) = O(n \log \log n) \).

10. Show using induction (the substitution method) that the recurrence

\[ T(n) = \begin{cases} 
2 \cdot T(n/2) + n \log n & \text{if } n > 2 \\
1 & \text{otherwise}
\end{cases} \]

has solution \( T(n) = O(n \log^2 n) \).

11. Show using induction (the substitution method) that the recurrence

\[ T(n) = \begin{cases} 
T(\sqrt{n}) + \log \log n & \text{if } n > 4 \\
1 & \text{otherwise}
\end{cases} \]

has solution \( T(n) = O((\log \log n)^2) \).

12. Show using induction (the substitution method) that the recurrence

\[ T(n) = \begin{cases} 
T(\sqrt{n}) + (\log \log n)^2 & \text{if } n > 4 \\
1 & \text{otherwise}
\end{cases} \]

has solution \( T(n) = O((\log \log n)^3) \).