Examining Initial Data with the Beetle-Burko Radiation Scalar: Analytical Examples

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Overview

- Motivation: ambiguities in construction of initial data
- The Beetle-Burko radiation scalar
  - Definition in terms of Weyl scalars
  - Construction from numerical relativity data
- Examples
  - Linear waves
  - Rotating black holes
- Discussion

Initial Data

- Decompose Einstein's equations for metric

$$ds^2 = g_{ab}dx^a dx^b = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

  into
  - evolution equations

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + 2D_{(i}\beta_{j)}$$

$$\partial_t K_{ij} = \ldots$$

  - constraint equations

$$R + K^2 - K_{ij}K^{ij} = -16\pi\rho \quad \text{(Hamiltonian constraint)}$$

$$D_i(K^{ij} - \gamma^{ij}K) = 8\pi j^j \quad \text{(Momentum constraint)}$$

- For initial data solve only constraint equations
  - four equations for twelve unknowns
  - Choice of decomposition and freely specifiable variables affects physical content of solution

    $\Longrightarrow$ "junk" gravitational radiation

- How can we measure gravitational wave content?

Decomposition of Weyl tensor

- Introduce complex null tetrad; e.g. from orthonormal basis $\mathbf{e}_t$, $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$

$$\mathbf{l} = \frac{1}{\sqrt{2}}(\mathbf{e}_t - \mathbf{e}_x), \quad \mathbf{k} = \frac{1}{\sqrt{2}}(\mathbf{e}_t + \mathbf{e}_x), \quad \mathbf{m} = \frac{1}{\sqrt{2}}(\mathbf{e}_y - i\mathbf{e}_z), \quad \bar{\mathbf{m}} = \frac{1}{\sqrt{2}}(\mathbf{e}_y + i\mathbf{e}_z)$$

  - satisfy $l_\alpha k^\alpha = -1$ and $m_\alpha \bar{m}^\alpha = 1$.
  - There are many such frames, related by null rotations, spin boosts and exchanges.

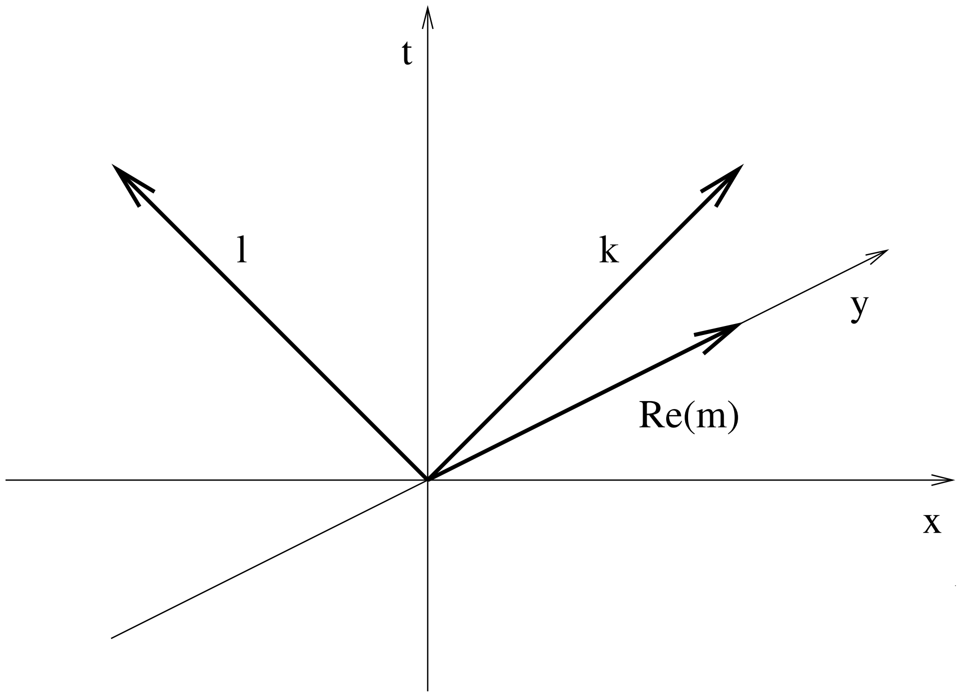
- Construct Weyl scalars $\psi_0, \cdots, \psi_4$

$$\psi_0 \equiv W_{\alpha\beta\gamma\delta} k^\alpha m^\beta k^\gamma m^\delta$$

and similar for others.

- In a *transverse* frame $\psi_1 = \psi_3 = 0$ and, in wave zone, can interprete

ψ_0	radiation along $\mathbf{l}$ ("ingoing")
ψ_2	"Coulomb" components
ψ_4	radiation along $\mathbf{k}$ ("outgoing")



- To extract gravitational radiation, would like to find ψ_0 and ψ_4 in transverse frame.

The Beetle-Burko radiation scalar

- Problem: transverse frames not unique
  - transversality condition $\psi_1 = \psi_3 = 0$ invariant under spin boosts
  - but ψ_0 and ψ_4 are not invariant
- But: Product $\psi_0 \psi_4$ invariant under spin boosts

  $\Longrightarrow$ define BB radiation scalar as

$$\boxed{\xi \equiv (\psi_0 \psi_4)^T}$$

  [Beetle & Burko (2002)]
- Remaining problem: in general there are three distinct classes of transverse frames

  $\Longrightarrow$ need to identify *quasi-Kinnersley* frame:
  - frame in which $\xi \to 0$ in limit of spacetime becoming algebraically special

  [Nerozzi *et.al.* (2005)]
- In generic spacetimes can interprete ξ in terms of radiation in radiation zones, and can extend definition into strong-field regions

  $\Longrightarrow$ may become useful tool for extraction of gravitational radiation [Beetle *et.al.* (2005); Nerozzi *et.al.* (2005)]

  $\Longrightarrow$ can use ξ to examine initial data sets

Computing the BB radiation scalar

- Compute electric and magnetic parts of Weyl tensor

$$\begin{aligned} E_{ij} &= \text{traceless part of } R_{ij} + KK_{ij} - K_{ik}K_j{}^k - 4\pi S_{ij} \\ B_{ij} &= \text{symmetric part of } -\epsilon_i{}^{kl}\nabla_k K_{lj} \end{aligned}$$

- Find complex tensor $C^i{}_j = E^i{}_j - iB^i{}_j$
- Compute scalar curvature invariants I and J

$$I = \frac{1}{2} C^i{}_j C^j{}_i \qquad\qquad J = -\frac{1}{6} C^i{}_j C^j{}_l C^l{}_i$$

- Compute speciality index $\mathcal{S}$

$$\mathcal{S} = 27\,\frac{J^2}{I^3}$$

- Compute BB radiation scalar

$$\xi = \frac{1}{4} I \left(2 - W(\mathcal{S})^{1/3} - W(\mathcal{S})^{-1/3}\right)$$

where $W(\mathcal{S}) = 2\mathcal{S} - 1 + 2\sqrt{\mathcal{S}(\mathcal{S}-1)}$
- Identify quasi-Kinnersley branch in asymptotic regime

[Beetle *et.al.* (2005)]

Linear Einstein-Rosen waves I

- Cylindrical wave with amplitude B and width a

$$ds^2 = -e^{-2\psi}dt^2 + e^{-2\psi}d\rho^2 + e^{2\psi}dz^2 + \rho^2 e^{-2\psi}d\phi^2$$

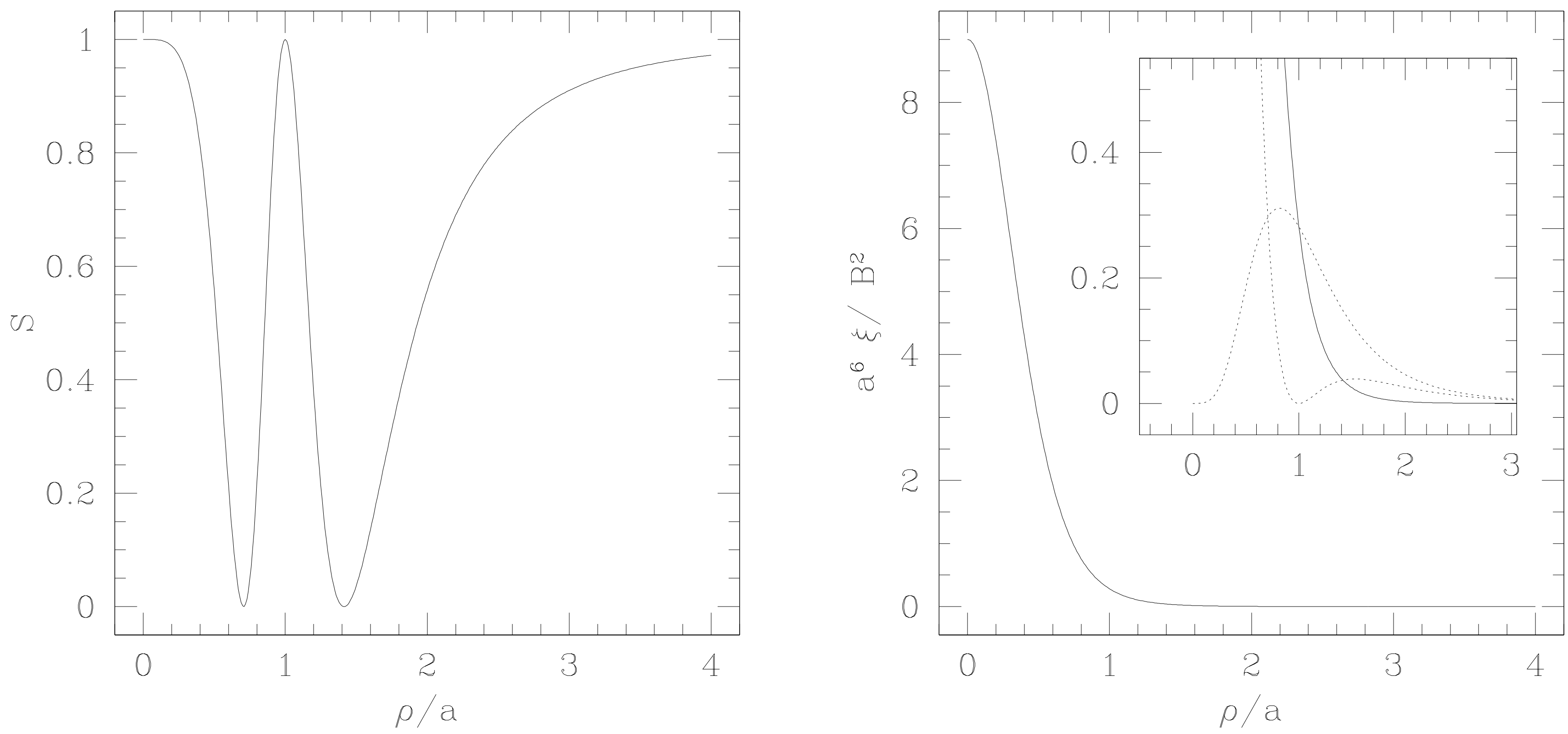
where

$$\psi(t,\rho) = B\left(\left((a+it)^2+\rho^2\right)^{-1/2} + \left((a-it)^2+\rho^2\right)^{-1/2}\right)$$

- identify α, $\beta^i = 0$ and γ_{ij}
- time symmetric about $t = 0$
  - $K_{ij} = 0$ at $t = 0$
  - greatly simplifies analysis:

$$E_{ij} = R_{ij}, \quad B_{ij} = 0$$

Linear Einstein-Rosen waves II



- $\mathcal{S}$ insensitive to wave amplitude B
- ξ scales with B^2
- Similar for Teukolsky waves

Rotating black holes

- Kerr black hole:
  - Petrov type D $\Longrightarrow$ algebraically special $\Longrightarrow$ $\mathcal{S} = 1$ $\Longrightarrow$ $\xi = 0$

- Bowen-York data:
  - assume conformal flatness $\bar{\gamma}_{ij} = \eta_{ij}$ and maximal slicing $K = 0$
  - analytical solution to momentum constraint

$$\bar{A}_{r\phi} = \frac{3 \sin^2 \theta}{r^2} L$$

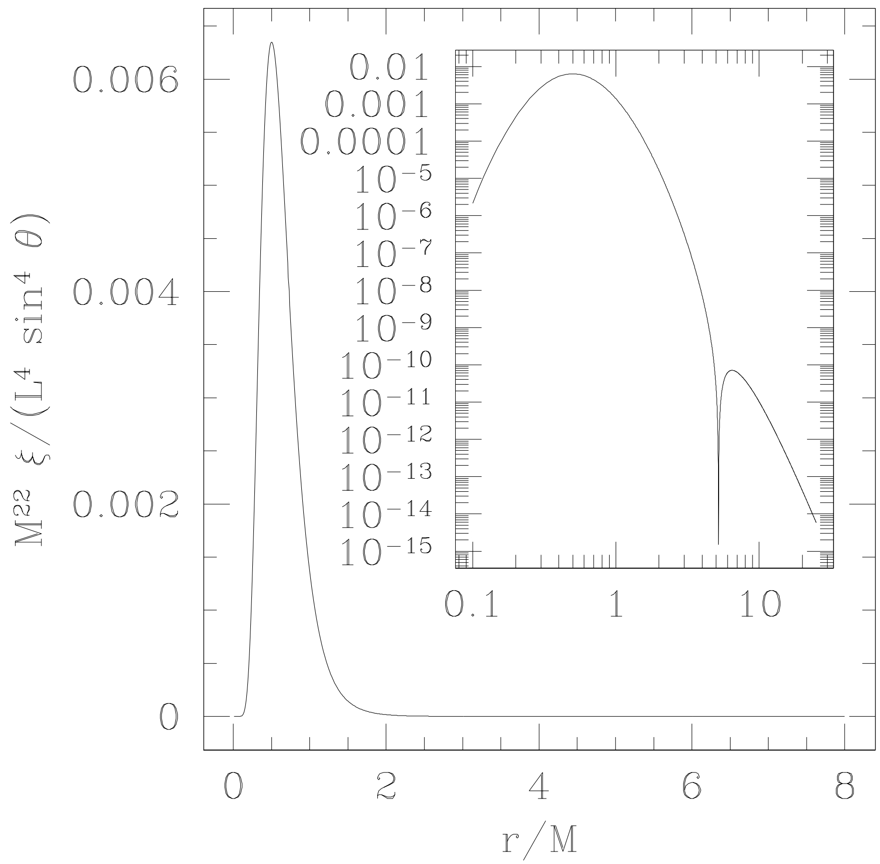
  [Bowen & York (1980)]
  - only need to solve Hamiltonian constraint

$$\bar{\nabla}^2 \psi = -\frac{1}{8} \psi^{-7} \bar{A}_{ij} \bar{A}^{ij}$$

  - usually solve numerically, but can construct analytical solution up to order L^2 (where deviations from Kerr first appear)
  [Gleiser *et.al.* (1998)]
  - Construct $\gamma_{ij} = \psi^4 \bar{\eta}_{ij}$ and $K_{ij} = \psi^{-2} \bar{A}_{ij}$

  $\Longrightarrow$ Compute ξ

Rotating black holes: Bowen-York solution



- Wave amplitude scales with L^2 [Gleiser *et.al.* (1998)]
- ξ scales with L^4 as expected

Discussion

- In generic spacetimes $\xi = (\psi_0 \psi_4)^T$ provides measure of radiation in radiation zones; can extend interpretation into strong-field regions

- Can use ξ to diagnose "gravitational wave content" of initial data sets
  - ξ scales with square of wave amplitude
  - picks up radiation content of Bowen-York data

- Potentially useful diagnostic for comparison of initial data sets obtained with different decompositions and/or different freely specifiable variables
  - may provide guidance for construction of astrophysically relevent initial data

Limitations

- For algebraically special spacetimes have $\mathcal{S} = 1$ and

$$\xi = 0$$

$\Longrightarrow$ can have spacetimes containing radiation, but $\xi = 0$ (e.g. plane waves)

- Can interprete ξ in terms of radiation only in radiation zones
$\Longrightarrow$ can have spacetimes without radiation, but $\xi \neq 0$ (e.g. cosmological solutions)

- No straightforward interpretation of $\xi = (\psi_0 \psi_4)^T$
  - Neither proportional to wave amplitude $(|\psi_0 + \psi_4|^2)$ nor wave power $(|\psi_0|^2 + |\psi_4|^2)$