Examining Initial Data with the Beetle-Burko Radiation Scalar: Analytical Examples

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Overview

• Motivation: ambiguities in construction of initial data

• The Beetle-Burko radiation scalar
  ○ Definition in terms of Weyl scalars
  ○ Construction from numerical relativity data

• Examples
  ○ Linear waves
  ○ Rotating black holes

• Discussion
Initial Data

• Decompose Einstein’s equations for metric

\[ ds^2 = g_{ab}dx^a dx^b = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \]

into

○ evolution equations

\[ \partial_t \gamma_{ij} = -2\alpha K_{ij} + 2D_{(i}\beta_{j)} \]

\[ \partial_t K_{ij} = ... \]

○ constraint equations

\[ R + K^2 - K_{ij}K^{ij} = -16\pi \rho \quad \text{(Hamiltonian constraint)} \]

\[ D_i(K^{ij} - \gamma^{ij} K) = 8\pi j^j \quad \text{(Momentum constraint)} \]

• For initial data solve only constraint equations

○ four equations for twelve unknowns

○ Choice of decomposition and freely specifiable variables affects physical content of solution

⇒ “junk” gravitational radiation

• How can we measure gravitational wave content?
Deomposition of Weyl tensor

- Introduce complex null tetrad; e.g. from orthonormal basis $\mathbf{e}_t$, $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$

$$\mathbf{l} = \frac{1}{\sqrt{2}}(\mathbf{e}_t - \mathbf{e}_x), \quad \mathbf{k} = \frac{1}{\sqrt{2}}(\mathbf{e}_t + \mathbf{e}_x), \quad \mathbf{m} = \frac{1}{\sqrt{2}}(\mathbf{e}_y - i\mathbf{e}_z), \quad \bar{\mathbf{m}} = \frac{1}{\sqrt{2}}(\mathbf{e}_y + i\mathbf{e}_z)$$

- Satisfy $l_\alpha k^\alpha = -1$ and $m_\alpha \bar{m}^\alpha = 1$.
- There are many such frames, related by null rotations, spin boosts and exchanges.

- Construct Weyl scalars $\psi_0, \ldots, \psi_4$

$$\psi_0 \equiv W_{\alpha\beta\gamma\delta} k^\alpha m^\beta \bar{k}^\gamma \bar{m}^\delta$$

and similar for others.

- In a transverse frame $\psi_1 = \psi_3 = 0$ and, in wave zone, can interpret:

- $\psi_0$ radiation along $\mathbf{l}$ ("ingoing")
- $\psi_2$ "Coulomb" components
- $\psi_4$ radiation along $\mathbf{k}$ ("outgoing")

- To extract gravitational radiation, would like to find $\psi_0$ and $\psi_4$ in transverse frame.
The Beetle-Burko radiation scalar

- Problem: transverse frames not unique
  - transversality condition $\psi_1 = \psi_3 = 0$ invariant under spin boosts
  - but $\psi_0$ and $\psi_4$ are not invariant
- But: Product $\psi_0\psi_4$ invariant under spin boosts
  $\Rightarrow$ define BB radiation scalar as
  $$\xi \equiv (\psi_0\psi_4)^T$$

[Beetle & Burko (2002)]

- Remaining problem: in general there are three distinct classes of transverse frames
  $\Rightarrow$ need to identify quasi-Kinnersley frame:
    - frame in which $\xi \to 0$ in limit of spacetime becoming algebraically special
[Nerozzi et.al. (2005)]

- In generic spacetimes can interpret $\xi$ in terms of radiation in radiation zones, and can extend definition into strong-field regions
  $\Rightarrow$ may become useful tool for extraction of gravitational radiation
  [Beetle et.al. (2005); Nerozzi et.al. (2005)]

$\Rightarrow$ can use $\xi$ to examine initial data sets
Computing the BB radiation scalar

- Compute electric and magnetic parts of Weyl tensor
  \[ E_{ij} = \text{traceless part of } R_{ij} + KK_{ij} - K_iK_j^\kappa - 4\pi S_{ij} \]
  \[ B_{ij} = \text{symmetric part of } -\epsilon_i^{kl}\nabla_kK_{lj} \]

- Find complex tensor \( C^i{}_j = E^i{}_j - iB^i{}_j \)

- Compute scalar curvature invariants \( I \) and \( J \)
  \[ I = \frac{1}{2} C^i{}_j C^j{}_i \]
  \[ J = -\frac{1}{6} C^i{}_j C^j{}_l C^l{}_i \]

- Compute speciality index \( S \)
  \[ S = 27 \frac{J^2}{I^3} \]

- Compute BB radiation scalar
  \[ \xi = \frac{1}{4} I \left( 2 - W(S)^{1/3} - W(S)^{-1/3} \right) \]
  where \( W(S) = 2S - 1 + 2\sqrt{S(S - 1)} \)

- Identify quasi-Kinnersley branch in asymptotic regime
  [Beetle et.al. (2005)]
**Linear Einstein-Rosen waves I**

- Cylindrical wave with amplitude $B$ and width $a$

$$ds^2 = -e^{-2\psi}dt^2 + e^{-2\psi}d\rho^2 + e^{2\psi}dz^2 + \rho^2 e^{-2\psi}d\phi^2$$

where

$$\psi(t, \rho) = B\left(\left((a + it)^2 + \rho^2\right)^{-1/2} + \left((a - it)^2 + \rho^2\right)^{-1/2}\right)$$

- identify $\alpha$, $\beta^i = 0$ and $\gamma_{ij}$

- time symmetric about $t = 0$
  - $K_{ij} = 0$ at $t = 0$
  - greatly simplifies analysis:

$$E_{ij} = R_{ij}, \quad B_{ij} = 0$$
• $S$ insensitive to wave amplitude $B$
• $\xi$ scales with $B^2$
• Similar for Teukolsky waves
Rotating black holes

• Kerr black hole:
  ◦ Petrov type D $\implies$ algebraically special $\implies S = 1 \implies \xi = 0$

• Bowen-York data:
  ◦ assume conformal flatness $\bar{\gamma}_{ij} = \eta_{ij}$ and maximal slicing $K = 0$
  ◦ analytical solution to momentum constraint
    $$\bar{A}_{r\phi} = \frac{3 \sin^2 \theta}{r^2} L$$
    [Bowen & York (1980)]
  ◦ only need to solve Hamiltonian constraint
    $$\bar{\nabla}^2 \psi = -\frac{1}{8} \psi^{-7} \bar{A}_{ij} \bar{A}^{ij}$$
  ◦ usually solve numerically, but can construct analytical solution up to order $L^2$
    (where deviations from Kerr first appear)
    [Gleiser et.al. (1998)]
  ◦ Construct $\gamma_{ij} = \psi^4 \bar{\eta}_{ij}$ and $K_{ij} = \psi^{-2} \bar{A}_{ij}$
  $\implies$ Compute $\xi$
Rotating black holes: Bowen-York solution

- Wave amplitude scales with $L^2$ [Gleiser et.al. (1998)]
- $\xi$ scales with $L^4$ as expected
Discussion

- In generic spacetimes \( \xi = (\psi_0 \psi_4)^T \) provides measure of radiation in radiation zones; can extend interpretation into strong-field regions.

- Can use \( \xi \) to diagnose “gravitational wave content” of initial data sets:
  - \( \xi \) scales with square of wave amplitude
  - picks up radiation content of Bowen-York data

- Potentially useful diagnostic for comparison of initial data sets obtained with different decompositions and/or different freely specifiable variables:
  - may provide guidance for construction of astrophysically relevant initial data
Limitations

- For algebraically special spacetimes have $\mathcal{S} = 1$ and
  \[ \xi = 0 \]
  \[ \implies \text{can have spacetimes containing radiation, but } \xi = 0 \text{ (e.g. plane waves)} \]

- Can interpret $\xi$ in terms of radiation only in radiation zones
  \[ \implies \text{can have spacetimes without radiation, but } \xi \neq 0 \text{ (e.g. cosmological solutions)} \]

- No straightforward interpretation of $\xi = (\psi_0 \psi_4)^T$
  - Neither proportional to wave amplitude $(|\psi_0 + \psi_4|^2)$ nor wave power $(|\psi_0|^2 + |\psi_4|^2)$