Numerical Simulations of Compact Binaries

Thomas Baumgarte

Bowdoin College

Lot's of recent progress in Numerical Relativity

- Initial value decompositions

- Evolution equations

- Relativistic hydrodynamics

$\Longrightarrow$ Significant improvements in simulations of neutron star and black hole binaries

3+1 decomposition

Einstein's equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

for metric

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

can be split into

- constraint equations (like "div" equations in E&M)

$$R + K^2 - K_{ij}K^{ij} = -16\pi\rho \qquad \text{(Hamiltonian constraint)}$$

$$D_i(K^{ij} - \gamma^{ij}K) = 8\pi j^j \qquad \text{(Momentum constraint)}$$

- evolution equations (like "curl" equations)

$$\frac{d}{dt}\gamma_{ij} \equiv \left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)\gamma_{ij} = -2\alpha K_{ij}$$

$$\frac{d}{dt}K_{ij} = -D_iD_j\alpha + \alpha(R_{ij} - 2K_{ik}K^k{}_j + KK_{ij} - 8\pi M_{ij})$$

[Arnowitt, Deser & Misner, 1962]

To construct initial data...

$\Longrightarrow$ ... solve constraint equations

To study evolution...

$\Longrightarrow$ ... solve evolution equations

Overview

- Initial data
  - Binary black hole initial data
  - Binary neutron stars initial data
- Evolution calculations
  - Binary black hole evolution simulations
  - Binary neutron star evolution simulations
- Black hole-neutron star binaries

Initial data:
conformal decomposition of constraint equations

Constraint equations provide fewer equations than unknowns

$\Longrightarrow$ decomposition of constraint equations singles out constrained from freely-specifiable quantities

$\Longrightarrow$ choice of decomposition and choice of freely-specifiable quantities affects physical content of solution

Two most popular decompositions:

- conformal transverse-traceless

- conformal thin-sandwich

Both conformally decompose metric

$$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$$

[Lichnerowicz, 1944; York, 1971, 1972]

and split extrinsic curvature into trace and traceless part

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K = \psi^{-10}\bar{A}^{ij} + \frac{1}{3}\gamma_{ij}K$$

$\Longrightarrow$ Differ in treatment of $\bar{A}^{ij}$...

Conformal transverse-traceless decomposition

Split $\bar{A}_{ij}$ into transverse-traceless and longitudinal parts

$$\bar{A}^{ij} = \bar{A}^{ij}_{\mathrm{TT}} + \bar{A}^{ij}_{\mathrm{L}}$$

where

$$\bar{D}_j \bar{A}^{ij}_{\mathrm{TT}} = 0$$

Insert into momentum constraint

$$\bar{D}_j \bar{A}^{ij} = \bar{D}_j \bar{A}^{ij}_{\mathrm{L}} = \frac{2}{3}\psi^6 \bar{\gamma}^{ij} \bar{D}_j K + 8\pi\psi^{10} j^i$$

$\Longrightarrow$ Momentum constraint becomes equation for longitudinal part $\bar{A}^{ij}_{\mathrm{L}}$

Constructing Binary Black Hole Initial Data
– conformal TT decomposition –

Choose freely specifiable data

$$\bar{\gamma}_{ij} = f_{ij} \qquad \text{(conformal flatness)}$$

$$K = 0 \qquad \text{(maximal slicing)}$$

$$\bar{A}^{ij}_{\mathrm{TT}} = 0$$

- Momentum constraint simplifies dramatically and decouples from Hamiltonian constraint [Bowen & York, 1980]
- Can construct analytical black hole solution $\bar{A}^{ij}_{\mathrm{L}}$ to momentum constraint
- Insert analytical $\bar{A}^{ij}$ into Hamiltonian constraint

$$\nabla^2\psi = -\frac{1}{8}\psi^{-7}\bar{A}_{ij}\bar{A}^{ij}$$

$\Longrightarrow$ binary black hole solution for given binary separation and angular momentum

- Subtleties:
  - ∘ singularities
  - ∘ circular orbit

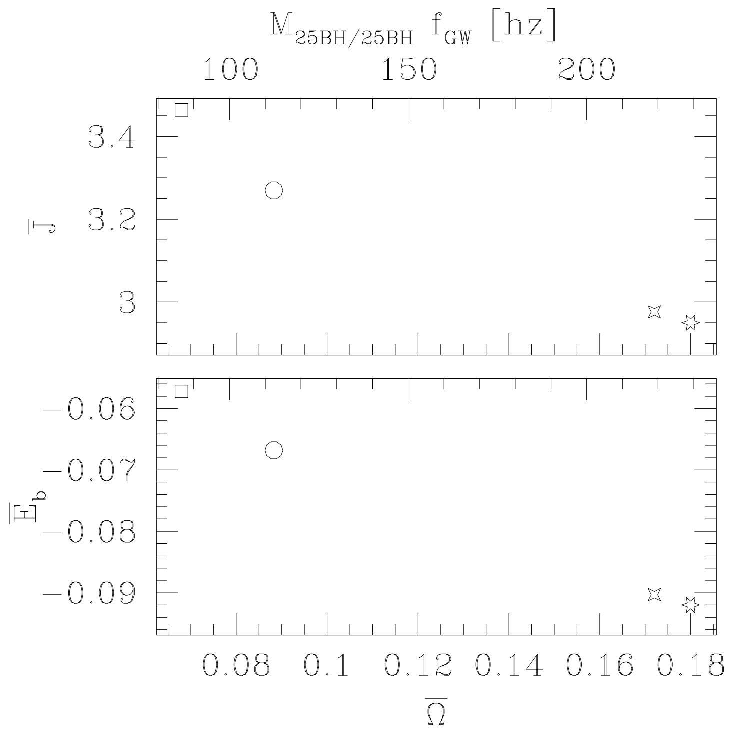
[Cook, 1994; Baumgarte, 2000; Tichy & Brügmann, 2003]

Comparison with post-Newtonian results I

Surprise: large disagreement with post-Newtonian results

[e.g. Damour, Jaranowski & Schäfer, 2000]

Values for Innermost Stable Circular Orbit (ISCO):



Conformal thin-sandwich decomposition

Consider traceless part of $\partial_t \gamma_{ij}$

$$u_{ij} \equiv \gamma^{1/3} \partial_t (\gamma^{-1/3} \gamma_{ij})$$

Then

$$u^{ij} = -2\alpha A^{ij} + (L\beta)^{ij}$$

or

$$\bar{A}^{ij} = \frac{\psi^6}{2\alpha} \left((\bar{L}\beta)^{ij} - \bar{u}^{ij} \right)$$

Insert into momentum constraint
$\Longrightarrow$ yields equation for shift β^i
Hamiltonian constraint
$\Longrightarrow$ equation for conformal factor ψ
[Isenberg, 1978; Wilson & Mathews, 1989; York 1999]

Constructing Binary Black Hole Initial Data
– conformal thin-sandwich decomposition –

For quasi-equilibrium data, it is natural to choose

$$\bar{u}^{ij} = 0$$

and

$$\partial_t K = 0$$

$\Longrightarrow$ equation for lapse α

Can also choose

$$\bar{\gamma}_{ij} = f_{ij} \quad \text{(conformal flatness)}$$
$$K = 0 \quad \text{(maximal slicing)}$$

Now have three coupled equations

$$\nabla_i (L\beta)^{ij} = \ldots \quad \text{momentum constraint}$$
$$\nabla^2 \psi = \ldots \quad \text{Hamiltonian constraint}$$
$$\nabla^2 \alpha = \ldots \quad \text{maximal slicing}$$

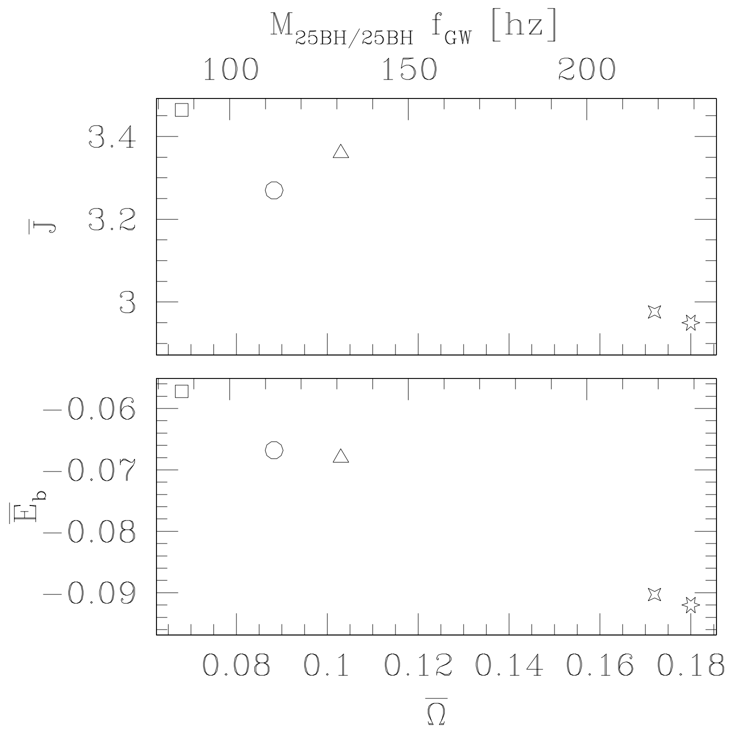
$\Longrightarrow$ binary black hole solution for given binary separation and angular momentum

[Grandclément, Gourgoulhon & Bonazzola, 2002]

Note: provides α and β^i together with initial data.

Comparison with post-Newtonian results II

Much better agreement with post-Newtonian results:



- Difference due to difference in decomposition of extrinsic curvature [Pfeiffer, Cook & Teukolsky, 2002; Damour, Gourgoulhon & Grandclément, 2002; Skoge & Baumgarte, 2002]
- Need better understanding of effect of freely specifiable quantities to construct binary black hole initial data that mimic "snapshot" of inspiral

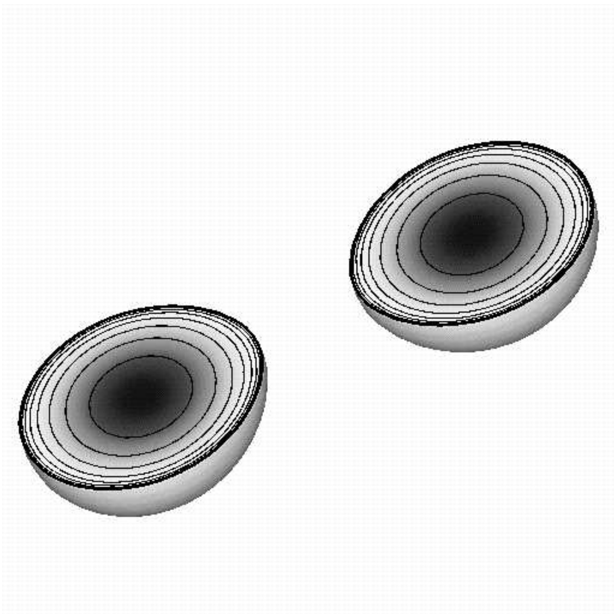
Binary neutron star initial data

- Need α and β^i for description of fluid four-velocity u^μ
  $\Longrightarrow$ adopt conformal thin-sandwich formalism

- For neutron stars in equilibrium relativistic Euler equations can be integrated once
  $\Longrightarrow$ for corotating binaries

$$\frac{h}{u^t} = \text{const}$$

[Baumgarte *et.al.*, 1997, 1998]



Irrotational binary neutron stars

Assume zero vorticity, then

$$hu_\alpha = \nabla_\alpha \phi$$

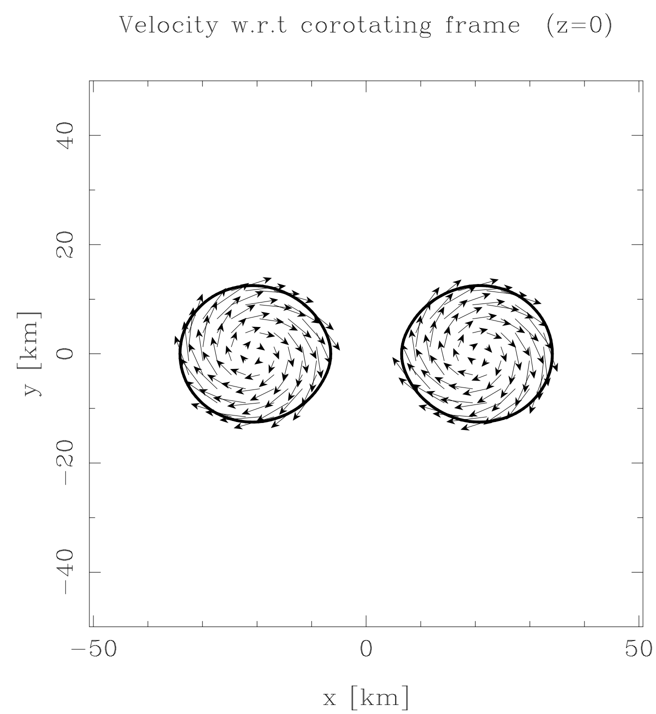
$\Longrightarrow$ first integral of Euler equation becomes elliptic equation for ϕ, boundary condition on surface of star

$\Longrightarrow$ harder, but more realistic than corotation

[Bonazzola, Gourgoulhon & Marck, 1997; Asada, 1998; Shibata, 1998; Teukolsky, 1998; Uryū & Eriguchi, 2000]

- Also: unequal mass binaries [Taniguchi & Gourgoulhon, 2002]

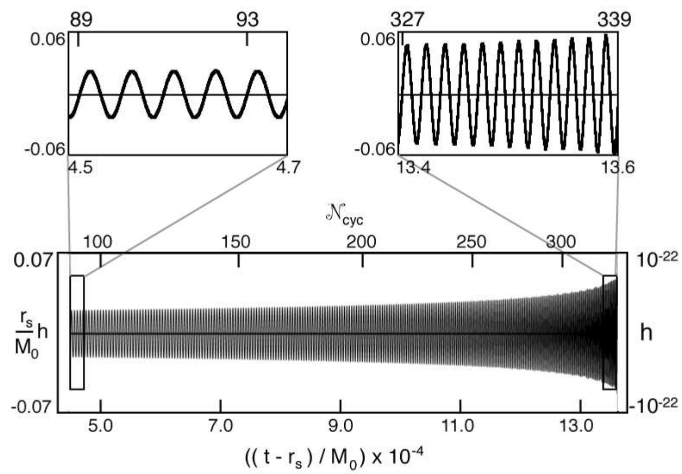
- Truly in equilibrium? [Miller & Suen, 2003]



[Gourgoulhon *et.al.*, 2001]

Inspiral sequences

Can construct "quasi-adiabatic" inspiral sequences [Duez *et.al.*, 2002]



- Inspiral rate can be computed from

$$\frac{dr}{dt} = \frac{dM/dt}{dM/dr}$$

$\Longrightarrow$ can construct gravitational wave train from inspiral phase

- Tidal elongation reduces central densities
- Sequences end at ISCO or tidal disruption [Taniguchi & Gourhoulhon, 2002]

Evolution Calculations

- Take initial data γ_{ij} and K_{ij}
- Evolve with evolution equations

$$\frac{d}{dt}\gamma_{ij} = -2\alpha K_{ij}$$

$$\frac{d}{dt}K_{ij} = -D_i D_j \alpha + \alpha(R_{ij} - 2K_{ik}K^k{}_j + KK_{ij} - 8\pi M_{ij})$$

Problem: Evolution of ADM equations leads to numerical instabilities even for small amplitude waves...

- Can re-write ADM equations in
  - explicitly hyperbolic form [e.g. Anderson & York, 1999; Kidder, Scheel & Teukolsky, 2001]
  - "BSSN" formulation [Shibata & Nakamura, 1995; Baumgarte & Shapiro, 1999]

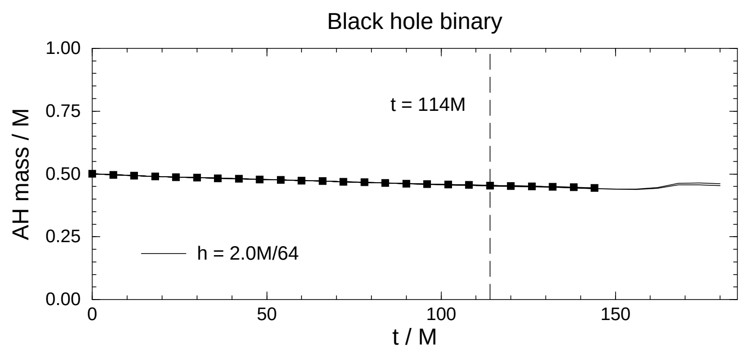
$\Longrightarrow$ significantly improves stability of numerical evolutions even for neutron stars and black holes

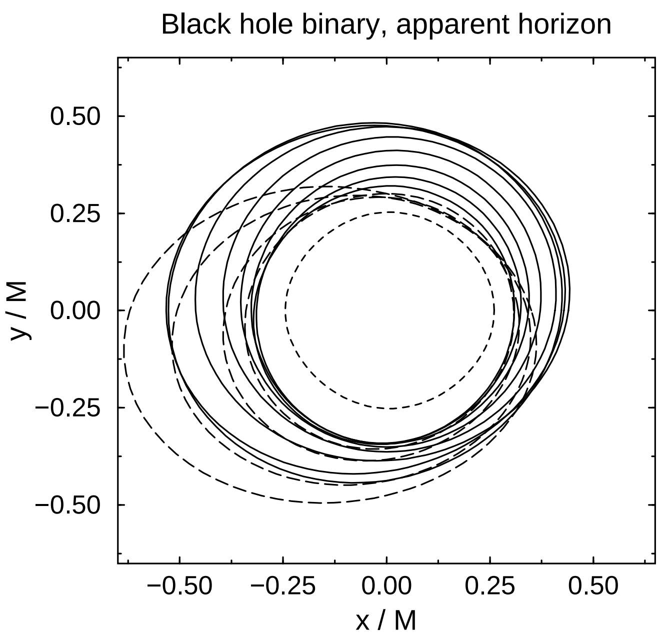
Subtleties...

- Coordinate Conditions
  - minimize coordinate-related time-variation in fields
  - computational efficiency
- Outer boundaries
  - location
  - typically "outgoing wave" boundary condition
  - inconsistent with Einstein's equations
  - gravitational wave extraction
- For black holes: singularity excision
  - inner boundary conditions?
  - "repopulation" of grid-points emerging from black hole
- Numerical implementation...

Binary Black Hole Evolution Calculations

First complete orbit of binary black hole system:





[Brügmann, Tichy & Jansen, 2004]

Fully relativistic hydrodynamics

See Font [2000] for review...

Can implement either...

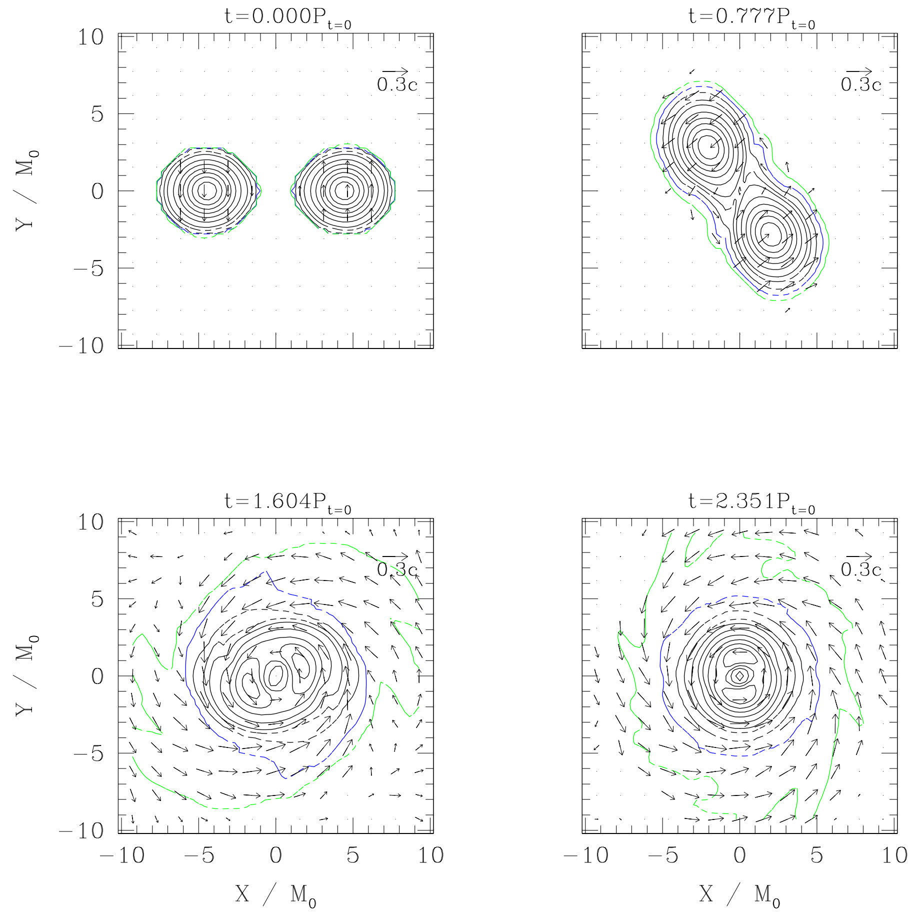
- "easy": artificial viscosity scheme
  [Shibata, 1999; Duez, Marronetti, Shapiro & Baumgarte, 2003]
- "harder": high-resolution shock-capturing schemes
  [Font *et.al.*, 2000, 2002]

Current "state of the art":

Simulations by Shibata, Taniguchi & Uryu [2004] ; evolve high-resolution shock-capturing scheme together with BSSN version of evolution equations for gravitational fields

Simulations of binary neutron star mergers

Example: irrotational binary, $n = 1$ polytrope, mass exceeds maximum allowed rest mass by 62%

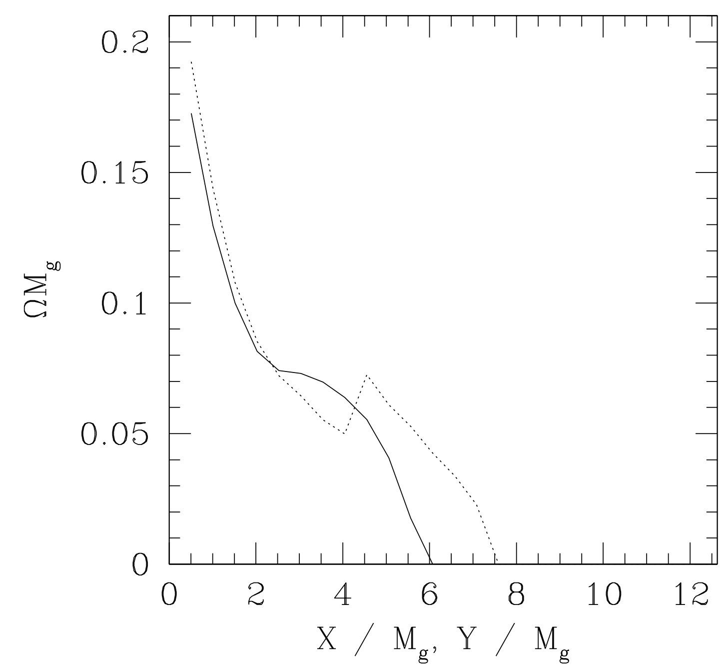


[Shibata, Taniguchi & Uryū, 2004]

$\Longrightarrow$ Surprising result: does *not* collapse to black hole

Effect of differential rotation

Remnant is *differentially* rotating:



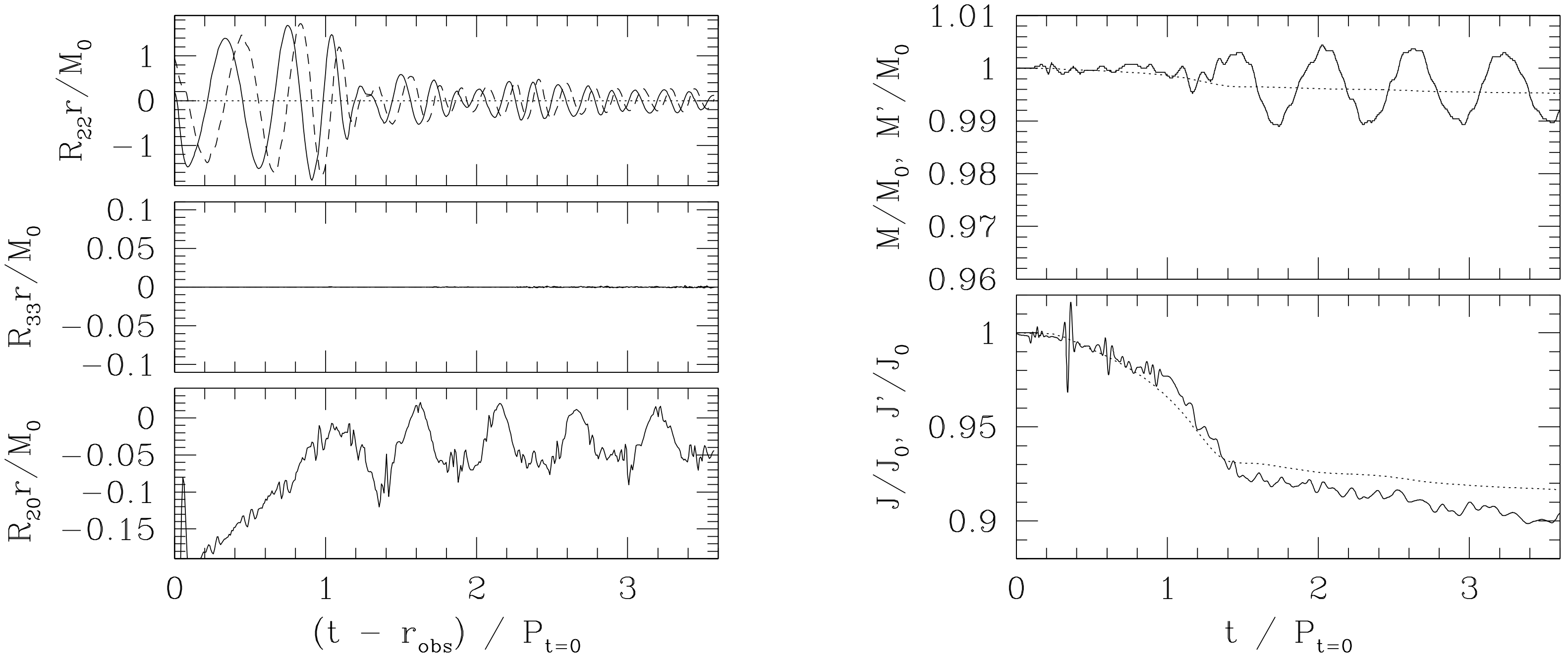
[Shibata & Uryū, 2000]

$\Longrightarrow$ Differential rotation can dramatically increase maximum allowed mass [Baumgarte, Shapiro & Shibata, 2000; Morrison, Baumgarte & Shapiro, 2004]

$\Longrightarrow$ binary neutron star coalescence may lead to "hypermassive" neutron star

Gravitational wave extraction

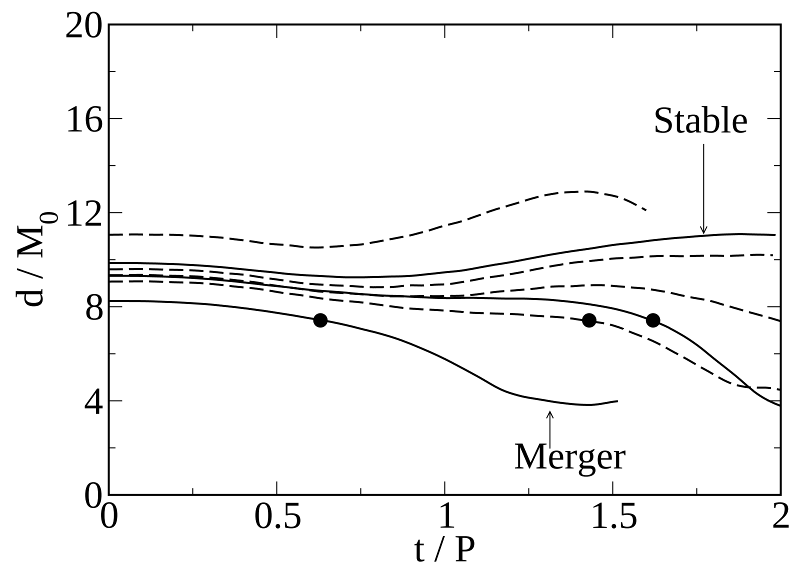
Extraction of gravitational wave forms and conservation of mass and angular momentum



[Shibata, Taniguchi & Uryū, 2004]

Dynamical determination of "ISCO"

Constant rest mass sequence with



[Marronetti, Duez, Shapiro & Baumgarte, 2004]

$\Longrightarrow$ Onset of dynamical instability fairly close to secular instability

Where are we?

- Binary neutron stars
  - both initial data and evolution simulations pretty well under control!
  - Things to be improved, but no show-stopper
- Binary black holes
  - can construct initial data, but need better understanding of astrophysical content
  - can evolve for limited time, but need to improve coordinate conditions/formulations/boundary conditions/numerical implementation...
- How about mixed black hole-neutron star binaries?

Black hole-neutron star binary initial data

For construction of initial data solve constraint equations in conformal thin-sandwich decomposition

- Adopt black hole solution (e.g. Kerr-Schild) as background
  $\Longrightarrow$ black hole companion

- Solve constraints and first integral of relativistic Euler equations
  $\Longrightarrow$ neutron star

As first step assume $M_{\rm BH} \gg M_{\rm NS}$

$\Longrightarrow$ construct constant rest-mass sequences

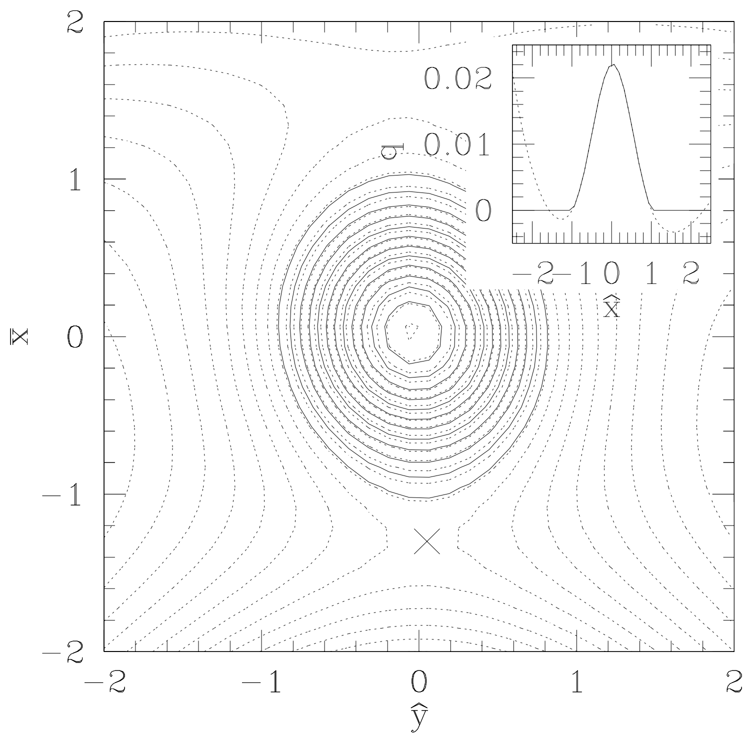
$\Longrightarrow$ locate onset of tidal disruption

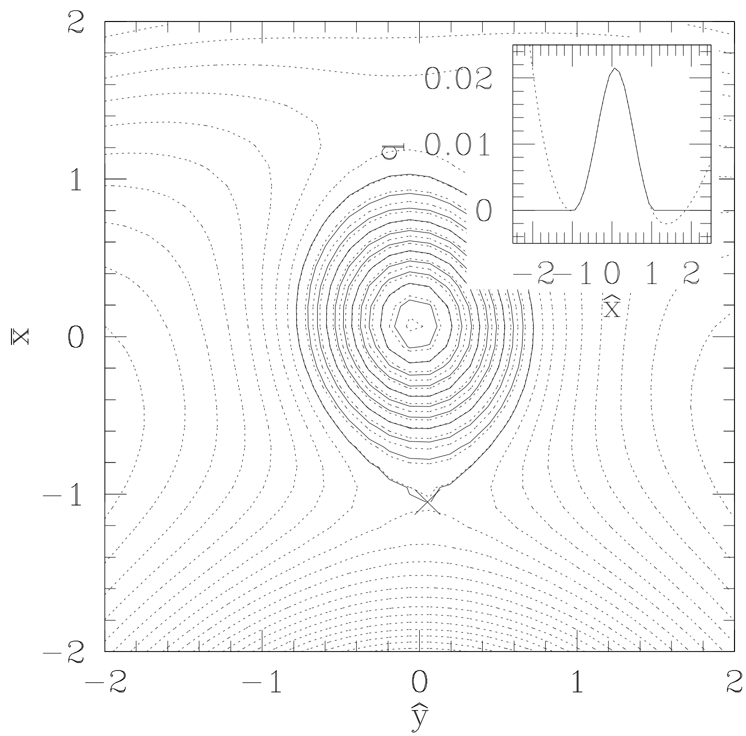
[Baumgarte, Skoge & Shapiro, submitted]

Location of tidal disruption

$n = 1$ polytrope fills out Roche lobe as defined from first integral of Euler equations

$\Longrightarrow$ onset of tidal disruption





$\hat{x}_{\rm BH} = -5.0$ $\qquad$ $M_{\rm BH} = 10 M_{\rm NS}$ $\qquad$ $\hat{x}_{\rm BH} = -4.18$

[Baumgarte, Skoge & Shapiro, submitted]

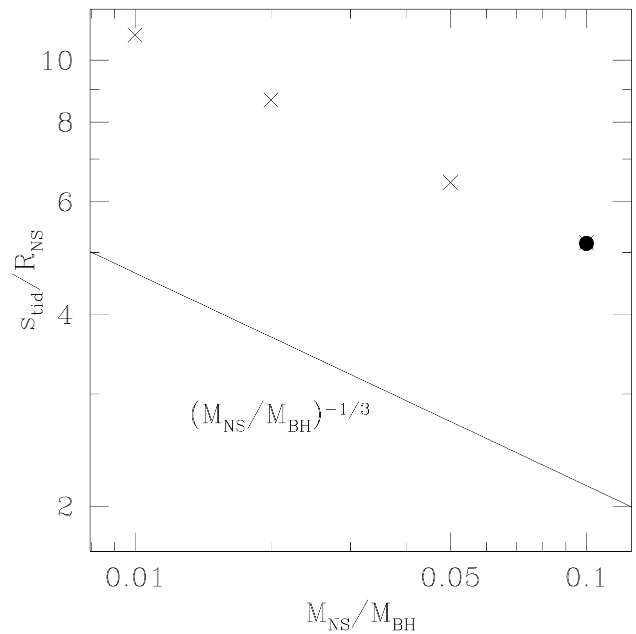
Scaling with mass ratio

Equate tidal force on test mass m on surface of neutron star with gravitational force exerted by star

$$F_{\rm tid} \sim G\frac{mM_{\rm BH}R_{\rm NS}}{s^3} \qquad F_{\rm grav} \sim G\frac{mM_{\rm NS}}{R_{\rm NS}^2}$$

$\Longrightarrow$ tidal disruption when

$$\frac{s_{\rm tid}}{R_{\rm NS}} \sim \left(\frac{M_{\rm BH}}{M_{\rm NS}}\right)^{1/3}$$



Here $R_{\rm NS} = 123 M_{\rm NS}$ - weakly relativistic regime

Summary

- Numerical relativity has seen rapid progress in recent years
- We can now handle neutron stars and (to some degree) black holes in three spatial dimensions
- Results are of astrophysical interest, and will hopefully aid in identification and interpretation of future detection of gravitational radiation