Black Hole-Neutron Star Binaries in General Relativity

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Why do we care?

Compact binaries (containing neutron stars and/or black holes) are promising sources of gravitational waves for LIGO, GEO and other gravitational wave detectors

Theoretical models are needed to help identify astrophysical sources in noisy output of detector

Theoretical Models of Compact Binaries

- Large Binary separation
  $\Longrightarrow$ post-Newtonian point-mass approximations

- Small Binary separation
  $\Longrightarrow$ numerical relativity:
    - initial data describing binary in quasicircular orbit
    - subsequent dynamical evolution: coalescence and merger/tidal disruption

- Binary black holes
  - initial data
    [Cook, 1994; Baumgarte, 2000; Gourgoulhon *et.al.*, 2002; ...]
  - dynamical evolution
    [e.g. Brügmann *et.al.*, 2004]
- Binary neutron stars
  - initial data
    [Baumgarte *et.al.*, 1997; Uryū *et.al.*, 2000; Taniguchi & Gourgoulhon, 2002]
  - dynamical evolution
    [Shibata & Uryū, 2000; Marronetti *et.al.*, 2004]
- Mixed black hole-neutron star (BHNS) binaries so far neglected
  [except possibly Miller 2001]

Onset of Tidal Disruption

Equate tidal force on test mass m on surface of neutron star

$$F_{\rm tid} \sim \frac{m M_{\rm BH} R_{\rm NS}}{s^3}$$

with gravitational force exerted by star

$$F_{\rm grav} \sim \frac{m M_{\rm NS}}{R_{\rm NS}^2}$$

$\Longrightarrow$ tidal disruption when

$$\frac{s_{\rm tid}}{R_{\rm NS}} \sim \left(\frac{M_{\rm BH}}{M_{\rm NS}}\right)^{1/3} \quad \text{or} \quad \frac{s_{\rm tid}}{M_{\rm BH}} \sim \left(\frac{M_{\rm NS}}{M_{\rm BH}}\right)^{2/3} \frac{R_{\rm NS}}{M_{\rm NS}}$$

For neutron star, $R_{\rm NS} \sim 5 M_{\rm NS}$, so

- for supermassive black holes $M_{\rm NS} \ll M_{\rm BH}$, so tidal effects not important
- for stellar mass black holes, $M_{\rm NS} \sim M_{\rm BH}$, so tidal disruption possible outside of Innermost Stable Circular Orbit (ISCO) at $s_{\rm ISCO} \sim 6 M_{\rm BH}$

Location of $s_{\rm tid}$ depends on equation of state (EOS)

$\Longrightarrow$ Observation of tidal disruption would provide wealth of astrophysical information [e.g. Vallisneri, 2000]

Overview: Quasiequilibrium Initial Data for BHNS Binaries

BHNS Binaries in Newtonian Gravity

Write Newtonian potential ϕ as

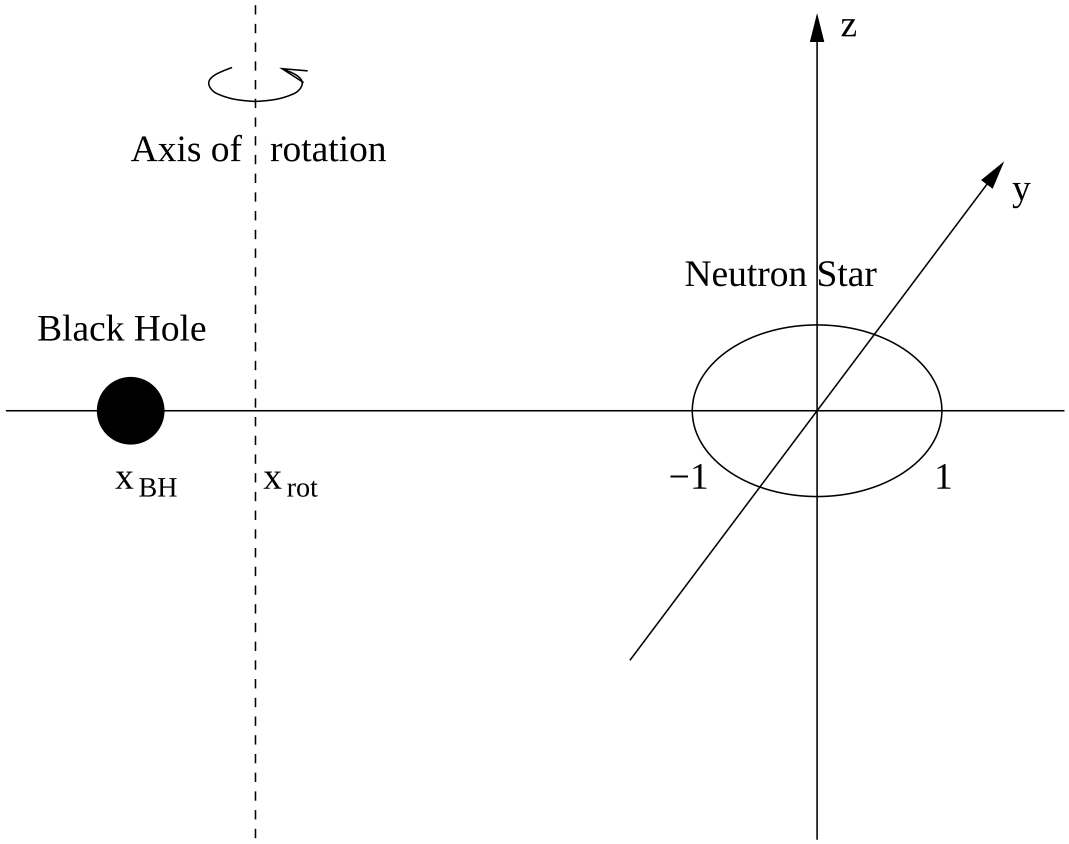
$$\phi = \phi_{\mathrm{BH}} + \phi_{\mathrm{NS}}$$

with

$$\phi_{\mathrm{BH}} = -\frac{M_{\mathrm{BH}}}{r_{\mathrm{BH}}}$$

Then Poisson equation

$$\boxed{\nabla^2\phi = \nabla^2\phi_{\mathrm{NS}} = 4\pi\rho}$$



Matter equation

Assume

- matter in equilibrium: stationary in corotating frame
- corotational fluid flow: $\mathbf{v} = \mathbf{\Omega} \times \mathbf{r}_{\rm rot}$
- polytropic equation of state

$$P = \kappa \rho^{1+1/n}$$

$\Longrightarrow$ can compute first integral of Euler equation .

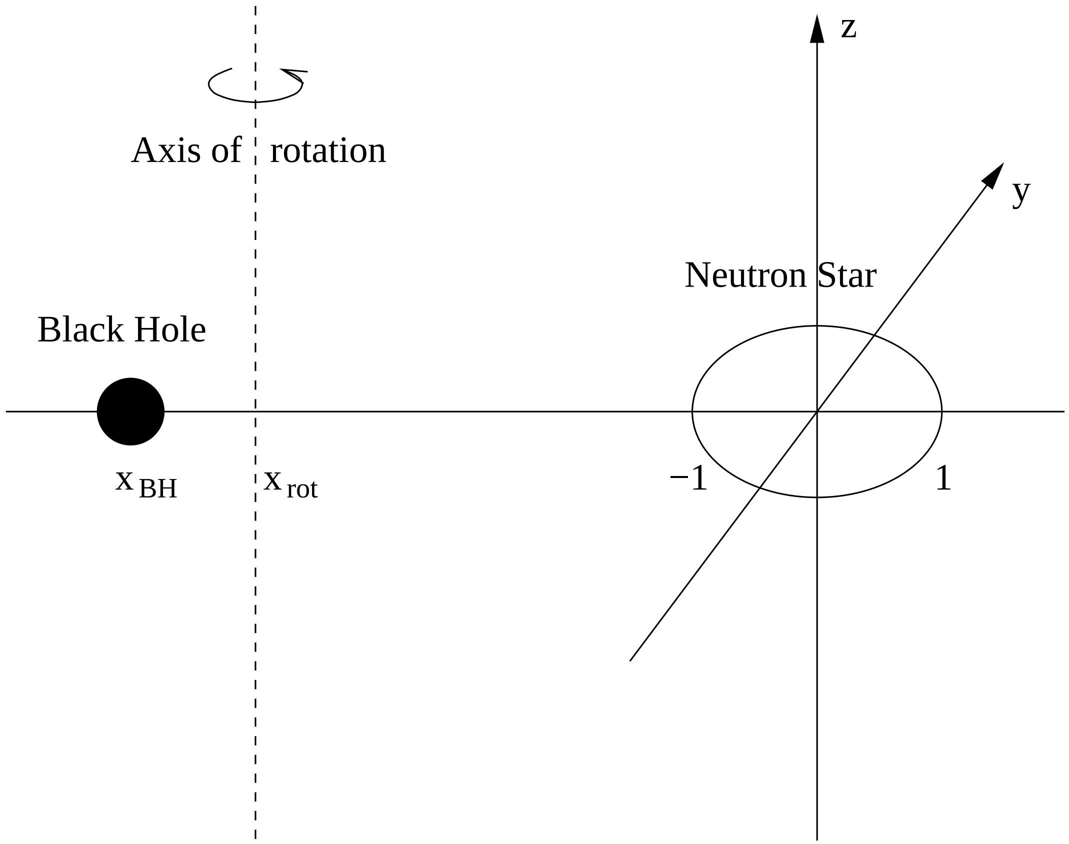
$$\boxed{q = \frac{1}{1+n}\left(C - \phi + \frac{1}{2}\Omega^2 \varpi_{\rm rot}^2\right) = \frac{1}{1+n}\left(C - \phi_{\rm eff}\right)}$$

where $q = P/\rho$

$\Longrightarrow$ have to solve Poisson equation together with algebraic equation for matter for ϕ, q and eigenvalues C and Ω

Extreme Mass Ratio

To simplify problem in first step, assume $M_{\rm BH} \gg M_{\rm NS}$



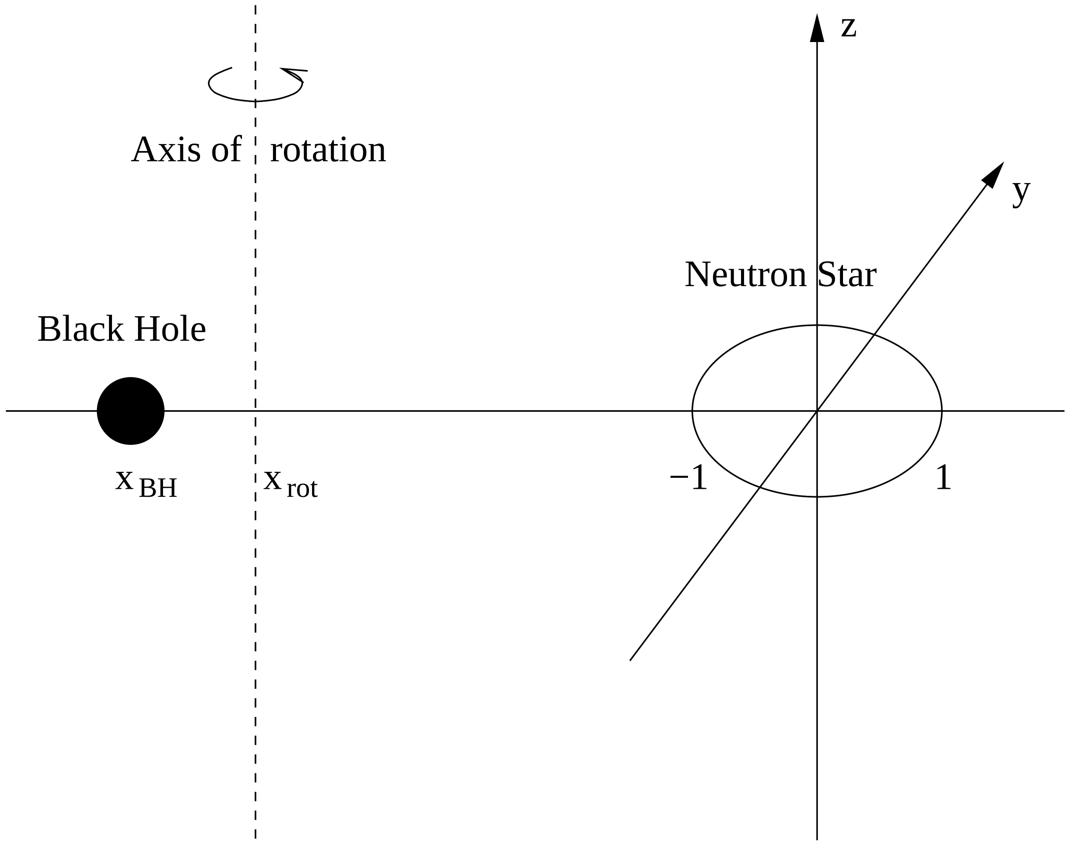
- neutron star affects spacetime only in its neighborhood, can therefore restrict numerical grid to this region

- no need to excise black hole [see Cook & Pfeiffer, 2004]

- no need to locate axis of rotation: $x_{\rm rot} = x_{\rm BH}$ (for example, compare ADM and Komar mass [Grandclément etal, 2002; Skoge & Baumgarte, 2002])

Numerical Strategy

- Location of stellar surface unknown a priori
- Rescale lengths with r_e so that surface intersects x-axis at -1 and 1
- Then three eigenvalues that have to be determined: Ω, C and r_e
- Can be determined by evaluating matter equation at $x = -1$, $x = 1$ and where matter density takes maximum value $q_{\max}$



Iteration

- To construct binary with fixed $q_{\max}$ and x_{BH}
  - solve Poisson equation
$$\nabla^2 \phi = \nabla^2 \phi_{\mathrm{NS}} = 4\pi q^n$$
  - find Eigenvalues Ω, C and r_e
  - solve integrated Euler equation
$$q = \frac{1}{1+n}\left(C - \phi + \frac{1}{2}\Omega^2 \varpi_{\mathrm{rot}}^2\right)$$
  - evaluate residuals of Poisson equation
- To construct binary with mixed M_{NS} iterate over $q_{\max}$
- To construct sequence vary binary separation x_{BH}

3+1 decomposition of Einstein's Equations

Einstein's equations

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

for metric

$$ds^2 = g_{ab}dx^a dx^b = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

can be split into

- constraint equations (like "div" equations in E&M)

$$R + K^2 - K_{ij}K^{ij} = -16\pi\rho \qquad \text{(Hamiltonian constraint)}$$

$$D_i(K^{ij} - \gamma^{ij}K) = 8\pi j^j \qquad \text{(Momentum constraint)}$$

- evolution equations (like "curl" equations)

$$\frac{d}{dt}\gamma_{ij} \equiv \left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)\gamma_{ij} = -2\alpha K_{ij}$$

$$\frac{d}{dt}K_{ij} = -D_i D_j \alpha + \alpha(R_{ij} - 2K_{ik}K^k{}_j + KK_{ij} - 8\pi M_{ij})$$

[Arnowitt, Deser & Misner, 1962]

To construct initial data...

⟹ ... solve constraint equations

To study evolution...

⟹ ... solve evolution equations

BHNS Binaries in General Relativity

- complete solution:
  - construct metric

$$ds^2 = g_{ab}dx^a dx^b = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

    that satisfies Einstein's field equations
  - stress energy tensor

$$T_{ab} = (\rho_0 + \rho_i + P)u_a u_b + P g_{ab}$$

    that satisfies relativistic equations of hydrodynamics (relativistic Euler equation)
- Quasiequilibrium initial data:
  - only solve constraint equations

$$R + K^2 - K_{ij}K^{ij} = -16\pi\rho \qquad \text{(Hamiltonian constraint)}$$

$$D_i(K^{ij} - \gamma^{ij}K) = 8\pi j^j \qquad \text{(Momentum constraint)}$$

  - natural to adopt conformal thin-sandwich decomposition
  - only solve integrated relativistic Euler equations

Conformal Decomposition of Hamiltonian Constraint

Conformally decompose metric

$$\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$$

and split extrinsic curvature into a trace and traceless part

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij} K = \psi^{-2} \bar{A}_{ij} + \frac{1}{3}\gamma_{ij} K$$

Then Hamiltonian constraint becomes

$$\boxed{\bar{D}^2\psi - \frac{1}{8}\psi \bar{R} - \frac{1}{12}\psi^5 K^2 + \frac{1}{8}\psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -2\pi\psi^5\rho}$$

[Lichnerowicz, 1944; York, 1971, 1972]

Conformal thin-sandwich decomposition

Consider traceless part of $\partial_t \gamma_{ij}$

$$u_{ij} \equiv \gamma^{1/3} \partial_t (\gamma^{-1/3} \gamma_{ij})$$

Then

$$u^{ij} = -2\alpha A^{ij} + (L\beta)^{ij}$$

(from evolution equation of γ_{ij}), or

$$\boxed{\bar{A}^{ij} = \frac{\psi^6}{2\alpha} \left((\bar{L}\beta)^{ij} - \bar{u}^{ij} \right)}$$

Insert into momentum constraint

$\Longrightarrow$ yields equation for shift β^i

$$\boxed{\bar{\Delta}_L \beta^i - (\bar{L}\beta)^{ij} \bar{D}_j \ln(\alpha \psi^{-6}) - \alpha \psi^{-6} \bar{D}_j (\alpha^{-1} \psi^6 \bar{u}^{ij}) - \frac{4}{3} \alpha \bar{D}^i K = 16\pi \alpha \psi^4 j^i}$$

[Isenberg, 1978; Wilson & Mathews, 1989; York 1999]

Freely specifiable variables: K, α, $\bar{\gamma}_{ij}$, $\bar{u}_{ij}$

Imposing Quasiequilibrium

For quasiequilibrium data in corotating coordinate system have

$$\bar{u}^{ij} = 0$$

Can also set

$$\partial_t K = 0$$

$\Longrightarrow$ equation for lapse α

$$\boxed{\bar{D}^2(\alpha\psi) = \frac{7}{8}\alpha\psi^{-7}\bar{A}_{ij}\bar{A}^{ij} + \frac{5}{12}\alpha\psi^5 K^2 + \frac{1}{8}\alpha\psi\bar{R} + \psi^5\beta^i\bar{D}_i K + 2\pi\alpha\psi^5(\rho + 2S)}$$

Remaining freely specifiable variables: K, $\bar{\gamma}_{ij}$

Summary Field equation

Have three coupled equations for gravitational fields ψ, α and β^i

$$\bar{D}^2\psi = ... \quad \text{(Hamiltonian constraint)}$$
$$\bar{\Delta}_L\beta^i = ... \quad \text{(momentum constraint)}$$
$$\bar{D}^2(\alpha\psi) = ... \quad (\text{from } \partial_t K = 0)$$

- Play role of Poisson equation in Newtonian gravity
- Still have to choose background $\bar{\gamma}_{ij}$ and K according to astrophysical situation at hand

Matter equation

Assume corotating fluid flow

$\Longrightarrow$ Equations of relativistic hydrodynamics can be integrated to yield

$$\frac{h}{u^t} = C + 1$$

- Find u^t from normalization $u_a u^a = -1$ (in corotating coordinates have $u^i = 0$)
- For polytropic equation of state

$$P = \kappa \rho_0^{1+1/n}$$

  can express enthalpy h and matter sources in field equations in terms of "density function" q

$\Longrightarrow$ integrated Euler equation becomes

$$\boxed{q = \frac{1}{1+n}\left(\frac{1+C}{(\alpha^2 - \psi^4 \bar{\gamma}_{ij}\beta^i\beta^j)^{1/2}} - 1\right)}$$

Compare with Newtonian cousin

$$q = \frac{1}{1+n}(C - \phi_{\rm eff})$$

$\Longrightarrow$ Can define relativistic effective potential

Constructing BHNS Binaries

- Choose black hole background solution (in Kerr-Schild coordinates)
  $\Longrightarrow$ black hole accounted for by background

- Solve equations in presence of equilibrium matter distribution
  $\Longrightarrow$ neutron star in presence of black hole

[Baumgarte, Skoge & Shapiro, 2004]

Kerr-Schild background

Compare Kerr-Schild metric

$$ds^2 = g_{ab}dx^a dx^b = (\eta_{ab} + 2Hl_a l_b)dx^a dx^b$$

where

$$H = \frac{M_{\rm BH}}{r_{\rm BH}}$$

with "ADM" metric

$$ds^2 = g_{ab}dx^a dx^b = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

to read off background quantities $\bar{\gamma}_{ij}$ and K as well as $\alpha_{\rm BH}$ and $\beta^i_{\rm BH}$.

- background satisfies equations in absence of neutron star with $\psi_{\rm BH} = 1$

Numerical Strategy

Split unknown field quantities into contributions from neutron star and black hole

$$\alpha = \alpha_{\rm NS} + \alpha_{\rm BH}$$

$$\psi = \psi_{\rm NS} + \psi_{\rm BH} = \psi_{\rm NS} + 1$$

$$\beta^i = \beta^i_{\rm NS} + \beta^i_{\rm BH} + \beta^i_{\rm rot} = \beta^i_{\rm NS} + \beta^i_{\rm BH} + (\vec{\Omega} \times \vec{r}_{\rm rot})^i$$

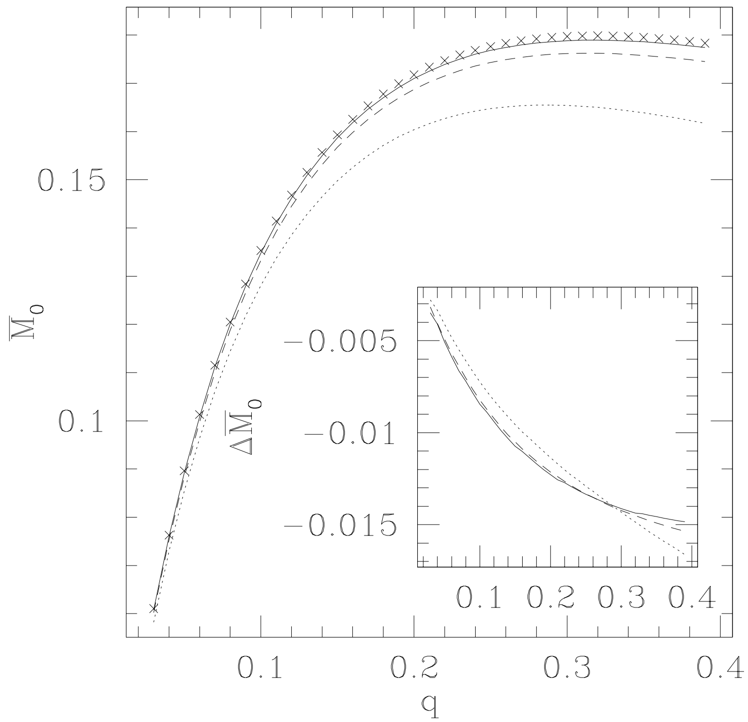
- Solve field equations for neutron star contributions $\alpha_{\rm NS}$, $\psi_{\rm NS}$ and $\beta^i_{\rm NS}$ together with integrated Euler equation for q
- Assume extreme mass ratios $M_{\rm BH} \gg M_{\rm NS}$
- Adopt rescaling and iteration identical to that in Newtonian problem
  $\Longrightarrow$ yields "eigenvalues" Ω, C and r_e together with solutions α, ψ, β^i and q

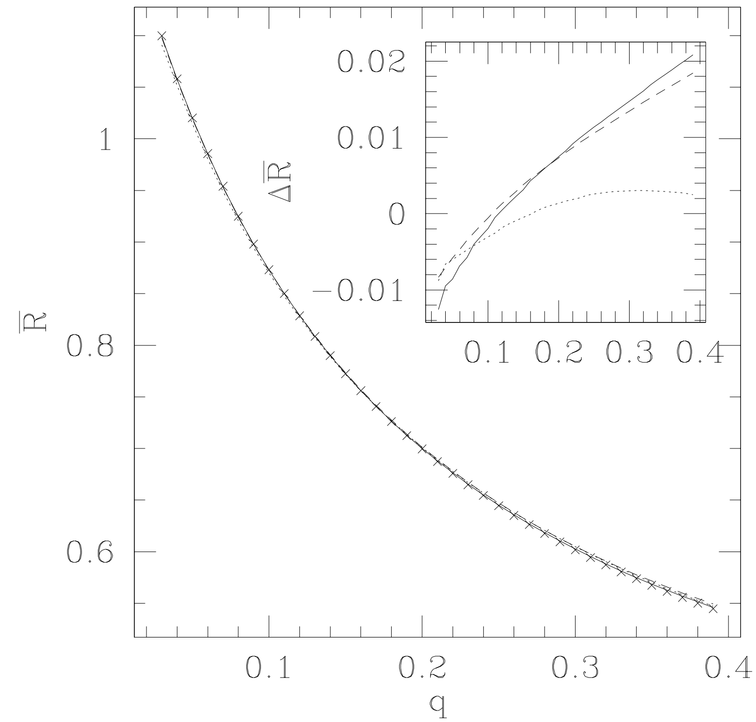
Neutron Stars in Isolation

Set $M_{\rm BH} = 0$

$\Longrightarrow$ Recover TOV solution for spherically symmetric neutron stars in isotropic coordinates

For $n = 1$ polytrope:

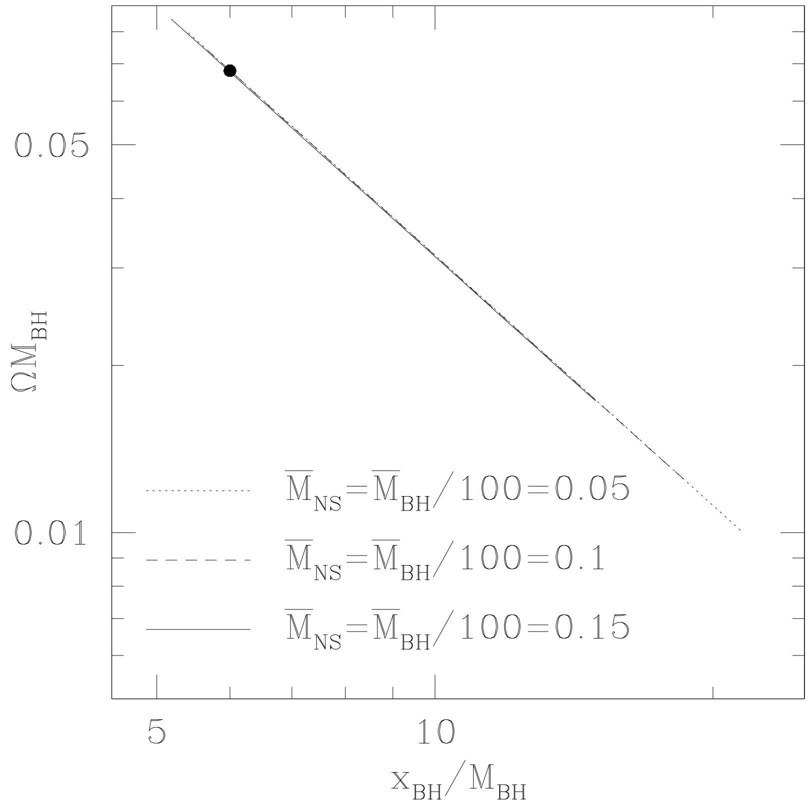




$\Longrightarrow$ 2nd order convergence to "analytical" solution

Constant Rest Mass Sequences

Construct constant rest-mass sequence



$\Longrightarrow$ recover Kepler's third law

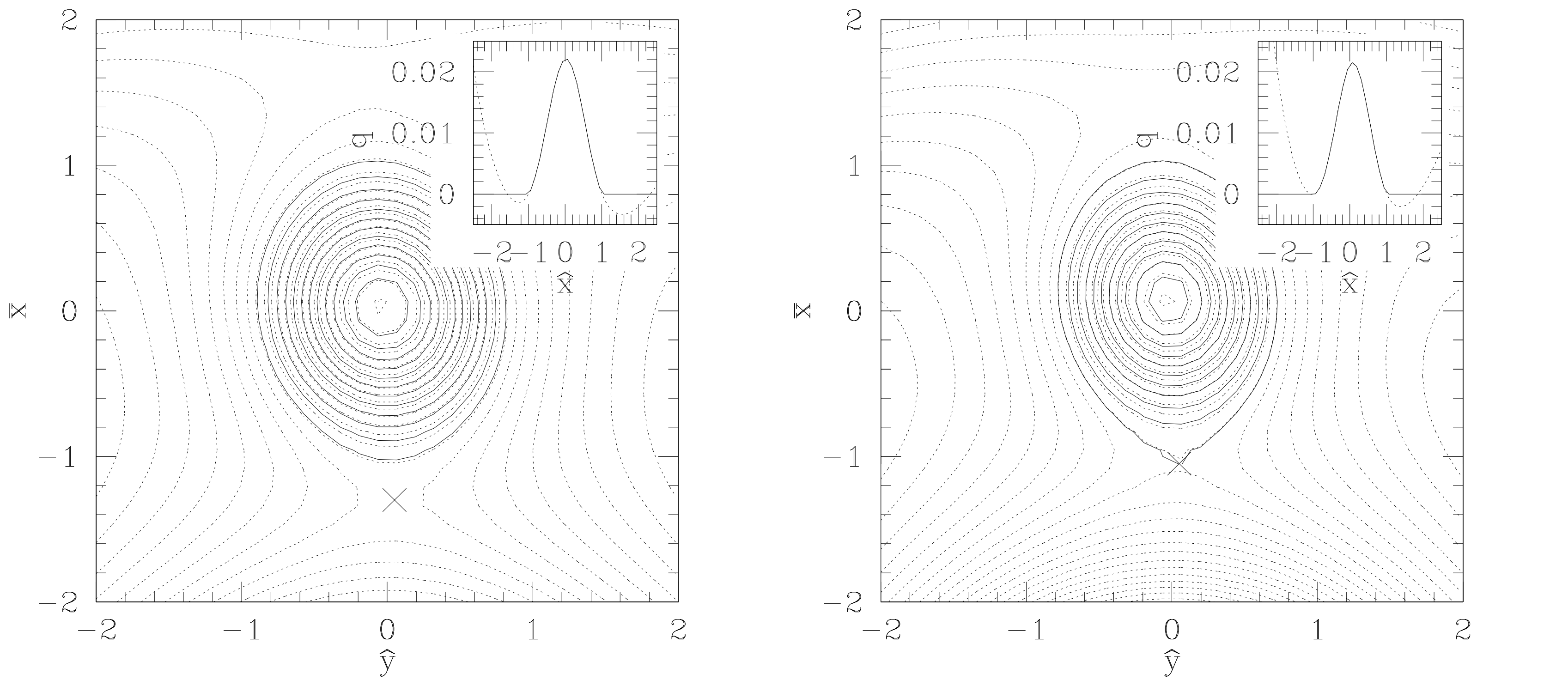
$$\Omega M_{\rm BH} = \left(\frac{x_{\rm BH}}{M_{\rm BH}}\right)^{-2/3}$$

independently of neutron star mass

Location of tidal disruption

$n = 1$ polytrope fills out Roche lobe as defined from first integral of Euler equations

$\Longrightarrow$ onset of tidal disruption



$\hat{x}_{\rm BH} = -5.0$ $\qquad$ $M_{\rm BH} = 10 M_{\rm NS}$ $\qquad$ $\hat{x}_{\rm BH} = -4.18$

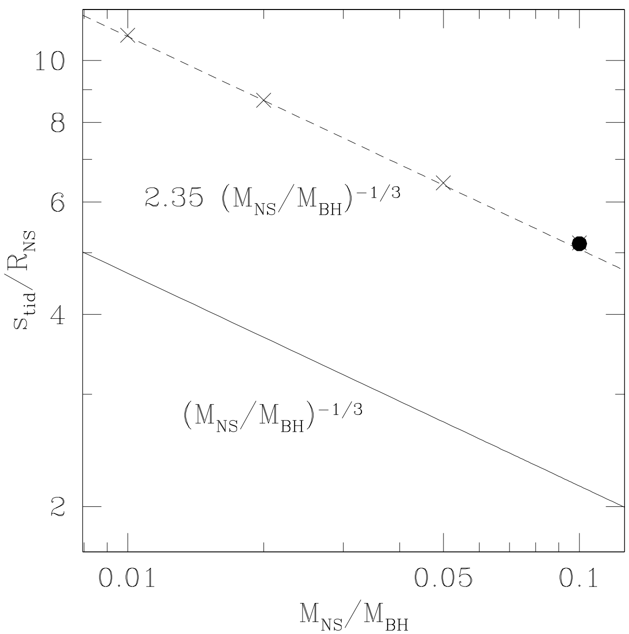
Scaling with mass ratio

Equate tidal force on test mass m on surface of neutron star with gravitational force exerted by star

$$F_{\rm tid} \sim G\frac{mM_{\rm BH}R_{\rm NS}}{s^3} \qquad F_{\rm grav} \sim G\frac{mM_{\rm NS}}{R_{\rm NS}^2}$$

$\Longrightarrow$ tidal disruption when

$$\frac{s_{\rm tid}}{R_{\rm NS}} \sim \left(\frac{M_{\rm BH}}{M_{\rm NS}}\right)^{1/3}$$



Here $R_{\rm NS} = 123M_{\rm NS}$ - weakly relativistic regime

Summary

- Have constructed relativistic models of black hole-neutron star binaries in quasiequilibrium
- Adopted extreme mass ratio limit $M_{\rm BH} \gg M_{\rm NS}$
- Located onset of tidal disruption
- In future...
  - ... relax assumption of extreme mass ratio
  - ... assume irrotational fluid flow
  - ... consider rotating black holes
  - ... study dynamical evolution