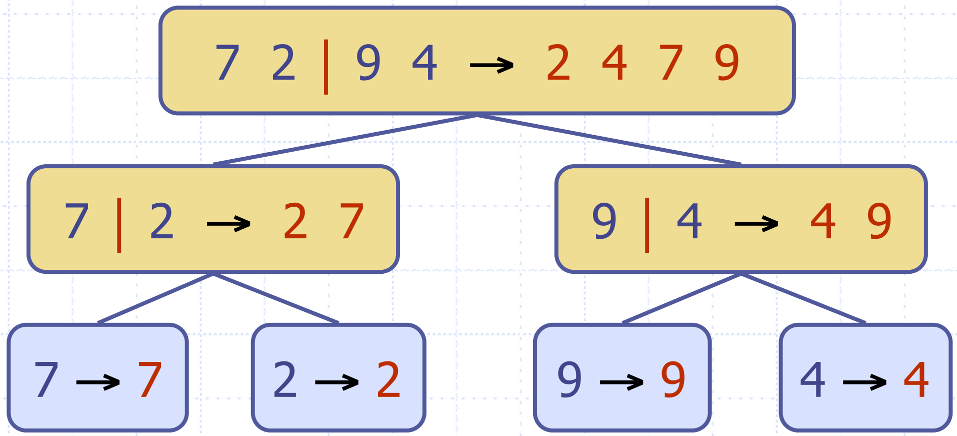
Recurrences



Outline

- Recurrence Equations
- Solving Recursion Equations
  - Recursion trees
  - Iterative substitution

Recurrence Equation Analysis

- The conquer step of merge-sort consists of merging two sorted sequences, each with $n/2$ elements and takes at most n steps, for some constant b, i.e. n steps
- The basis case ($n < 2$) will take 1 step.
- If we let $T(n)$ denote the running time of merge-sort on n items:

$$T(n) = 2T(n/2) + n$$

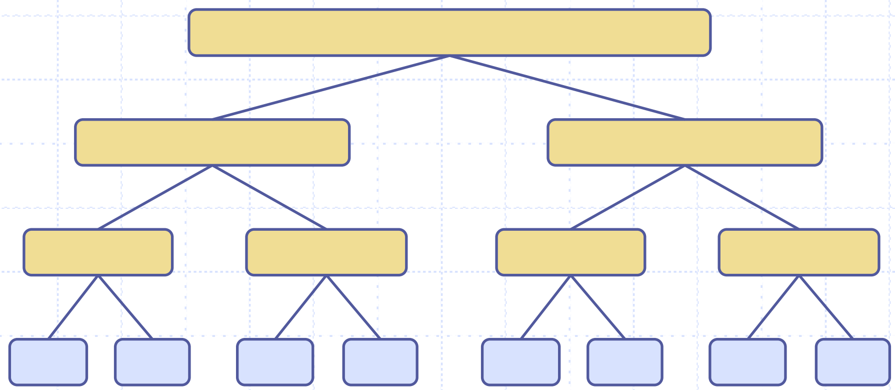
- We analyze the running time of merge-sort by finding a **closed form solution** to the above equation, i.e. a solution that has $T(n)$ only on the left-hand side.

Recursion Tree Method

- Draw the recursion tree for the recurrence relation and look for a pattern and then try to prove it is true by induction:

$$T(n) = 2T(n/2) + n$$

T's	size		time
1	n		n
2	$n/2$		n
…	…		n
2^i	$n/2^i$		…



Total time = $n \lg n$

Iterative Substitution Method

- In the iterative substitution technique, we iteratively apply the recurrence equation to itself and see if we can find a pattern.

$$\begin{aligned} T(n) &= 2T(n/2) + n \\ &= 2(2T(n/4) + n/2) + n \\ &= 4T(n/4) + 2n \\ &= 8T(n/8) + 3n \\ &= 2^4 T(n/2^4) + 4n \\ &= \ldots \\ &= 2^i T(n/2^i) + in \end{aligned}$$

- We reach the end of the recursion when $2^i = n$. That is, i = lg n.
- So, $T(n) = n + n\lg n$
- Thus, T(n) is O(n lg n).

Should Prove by Induction

- Parallels the recursion process
- We won’t do that. ☹

Master Method

- Many divide-and-conquer recurrence equations have the form:

$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \ge d \end{cases}$$

- The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a-\varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a}\log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a}\log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a+\varepsilon})$, then $T(n)$ is $\Theta(f(n))$,

   provided $af(n/b) \le \delta f(n)$ for some $\delta < 1$.