P-ADIC L-FUNCTIONS

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1. Introduction

The purpose of this paper is to construct the p-adic zeta function and p-adic Dirichlet L-functions. I studied Koblitz [6, Chapter 2] and Iwasawa [5], and introduce their approaches of the constructions.

The first section consists of the properties of the Riemann zeta function. One of the goal in this section is to prove the functional equation of the completed zeta function, $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ where Γ is the gamma function. Using the result and some of the properties of the gamma function, the values of $\zeta(s)$ are also expressed in term of the Bernoulli numbers for all the negative integers and all the even positive integers.

Section 2 and Section 3 are based on the study of Koblitz [6, Chapter 2]. Section 2 gives us the preparation for interpolating the Riemann zeta function p-adically. All the tools that we are going to use for constructing the p-adic zeta function, such as p-adic distribution and the integration, are introduced here. The interesting and very useful example of p-adic distribution is the k-th Bernoulli distribution, given by $\mu_k(a+p^N\mathrm{Z}_p) = p^{N(k-1)}B_k(a/p^N)$, where $B_k(x)$ is the k-th Bernoulli polynomial. The measure obtained by making a modification on μ_k, namely $\mu_{k,\alpha}(U) = \mu_k(U) - \alpha^{-k}\mu_k(\alpha U)$ with some α, is going to have an important role in Section 3. Using this measure with $k = 1$, the p-adic zeta function on the set of negative integers is defined as $\zeta_p(1-k) = (\alpha^{-k}-1)^{-1}\int_{\mathrm{Z}_p^\times} x^{k-1}\mu_{1,\alpha}$. Some of the properties in Section 3 will show that this actually differs from $\zeta(1-k)$ only by a factor of $(1-p^{k-1})$. Also, I would like to note that Kummer congruence shows that $\zeta_p(1-k)$ is a p-adic integer and that $\zeta_p(1-k)$ is continuous for $k \in \mathrm{Z}$, $k \geq 0$. (See Theorem 4.4) Our goal in this section is to extend the definition to Z_p.

The Dirichlet L-functions and its p-adic interpolation are discussed in later sections. In particular, all the properties of the zeta function that are discussed in earlier section are extended to the L-functions. The construction of p-adic L-functions are based on the study of Iwasawa [5]. One of the key ingredients is a Dirichlet character defined as $\chi_n := \chi\cdot\omega^{-n}$ where χ is a primitive Dirichlet character and $\omega(a)$ is the Teichmüler representative of a p-adic unit a. Iwasawa's approach does not involve the distribution. Instead, he constructed a power series that has all the desired properties, namely $A_\chi(x) = \sum_{n=0}^{\infty} c_n\binom{x}{n}$ where $c_n = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}(1-\chi_i(p)p^{i-1})B_{i,\chi_i}$. Using this series, the p-adic L-functions are defined as $L_p(s,\chi) = (s-1)^{-1}A_\chi(1-s)$.

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2. The Zeta Function

In this section, we consider the Riemann zeta function, $\zeta(s) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}$, $s \in \mathrm{C}$, and review some of the properties.

Proposition 2.1. *$\zeta(s) = \displaystyle\sum_{n=1}^{\infty}(1/n^s)$ converges absolutely for $Re(s) > 1$ and satisfies the Euler identity*

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}},$$

where p runs through all the primes.

Proof. For $Re(s) \geq 1+\delta$, we have $\sum_{n=1}^{\infty} |1/n^s| \leq \sum_{n=1}^{\infty} \left(1/n^{1+\delta}\right) < \infty$. So it converges absolutely.
Now we wish to show the Euler identity holds. First, we will show the absolute convergence of $\prod_p 1/(1-p^{-s})$ where $Re(s) \geq 1+\delta$. Note that, by the definition of the convergence of an infinite product, we need to show that $\displaystyle\sum_p \log\left(1-p^{-s}\right)^{-1}$ converges absolutely, where log is the principal branch of the logarithm.

$$\sum_p \left(-\log(1-p^{-s})\right) = \sum_p \left(-\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(-p^{-s})^n\right) = \sum_p \sum_{n=1}^{\infty} \frac{1}{np^{ns}}.$$

If $Re(s) \geq 1+\delta$,

$$\begin{aligned}
\sum_p \sum_{n=1}^{\infty} \frac{1}{n\,|p^{ns}|} &\leq \sum_p \sum_{n=1}^{\infty} \frac{1}{np^{n(1+\delta)}} \leq \sum_p \sum_{n=1}^{\infty} \left(\frac{1}{p^{1+\delta}}\right)^n \\
&= \sum_p \frac{1}{p^{1+\delta}-1} \leq 2\sum_p \frac{1}{p^{1+\delta}} < \infty.
\end{aligned}$$

Thus, $\sum_p(-\log(1-p^{-s}))$ converges absolutely, and so does $\prod_p 1/(1-p^{-s})$.

Now, let $\epsilon > 0$, and choose N such that $\displaystyle\sum_{n>N} \frac{1}{n^{1+\delta}} < \epsilon$. Let $\{p_i\}_{i=1}^{r}$ be the set of all the prime numbers less than or equal to N. Then

$$\begin{aligned}
\prod_{p\leq N} \frac{1}{1-p^{-s}} &= \prod_{p\leq N} \sum_{n=0}^{\infty} \frac{1}{p^{ns}} \\
&= 1 + \sum_{1\leq i\leq r} \frac{1}{p_i^s} + \sum_{1\leq i\leq j\leq r} \frac{1}{p_i^s p_j^s} + \sum_{1\leq i\leq j\leq k\leq r} \frac{1}{p_i^s p_j^s p_k^s} + \cdots \\
&= \sum_{n=\prod_i p_i^{\nu_i}} \frac{1}{n^s} \quad \text{by the Fundamental Theorem of Arithmetic} \\
&= \sum_{n\leq N} \frac{1}{n^s} + \sum_{\substack{n>N \\ n=\prod_i p_i^{\nu_i}}} \frac{1}{n^s}.
\end{aligned}$$

Therefore, we obtain

$$\left|\prod_{p\le N}\frac{1}{1-p^{-s}}-\zeta(s)\right|=\left|\sum_{\substack{n>N\\ n\neq\prod_i p_i^{\nu_i}}}\frac{1}{n^s}\right|\le\sum_{n>N}\frac{1}{n^{1+\delta}}<\epsilon,$$

and this completes the proof of the Euler identity. □

Theorem 2.2. *Let* $\Lambda(s):=\pi^{-s/2}\Gamma(s/2)\zeta(s)$, *where* Γ *is the gamma function. Then* Λ *is called the completed zeta function and satisfies the functional equation*

$$\Lambda(s)=\Lambda(1-s).$$

Proof. Let $F(z)=\dfrac{ze^z}{e^z-1}$, $G(z)=F(-z)z^{-1}$, and $H(s)=\displaystyle\int_{C^-}F(z)z^{s-1}\frac{dz}{z}$, where $C^-=(-\infty,-\epsilon]+\{\gamma^+(t):=\epsilon e^{it}:t\in[-\pi,\pi]\}+[-\epsilon,-\infty)$. Note that $G(z)=e^{-z}/(1-e^{-z})=1/(1-e^{-z})-1=\sum_{n=1}^{\infty}e^{-nz}$.
Now, if we let $z\mapsto -z$ in $H(s)$, since $(-z)^{s-1}=e^{(s-1)(\log|-z|+i(\arg z-\pi))}=-e^{\pi is}z^{s-1}$, we have

$$H(s)=-e^{\pi is}\int_{C^+}F(-z)z^{s-1}\frac{dz}{z}=-e^{\pi is}\int_{C^+}G(z)z^{s-1}dz,$$

with $C^+=(\infty,\epsilon)+\{\gamma^-(t):=\epsilon e^{-it}:t\in[0,2\pi]\}+(\epsilon,\infty)$. Let us consider

$$\int_{C^+}G(z)z^{s-1}dz=\int_{(\infty,\epsilon)}G(z)z^{s-1}dz+\int_{\gamma^-}G(z)z^{s-1}dz+\int_{(\epsilon,\infty)_{2\pi}}G(z)z^{s-1}dz.$$

The first integral equals $-\displaystyle\int_\epsilon^\infty G(t)t^{s-1}dt$, and the third integral is $e^{2\pi is}\displaystyle\int_\epsilon^\infty G(t)t^{s-1}dz$ because $z^{s-1}=e^{(s-1)\log|z|}e^{(s-1)2\pi i}$. For the second integral, one has

$$\int_{\gamma^-}G(z)z^{s-1}dz=-i\int_0^{2\pi}G(\epsilon e^{-it})\epsilon^s e^{-its}dt.$$

Therefore, for $Re(s)>1$, the integral approaches zero as $\epsilon\to 0$ since $G(\epsilon e^{-it})\epsilon^s$ tends to zero as $\epsilon\to 0$. Taking $\epsilon\to 0$, one has

$$\int_{C^+}G(z)z^{s-1}dz=(e^{2\pi is}-1)\int_0^\infty G(t)t^{s-1}dt,$$

and so

$$\begin{aligned}
H(s) &= (-e^{\pi is}+e^{-\pi is})\int_0^\infty G(t)t^{s-1}dt\\
&= -2i\sin(\pi s)\sum_{n=1}^{\infty}\int_0^\infty e^{-nt}t^{s-1}dt\\
&= -2i\sin(\pi s)\sum_{n=1}^{\infty}\frac{1}{n^s}\int_0^\infty e^{-nt}(nt)^s\frac{dt}{t}=-2i\sin(\pi s)\zeta(s)\Gamma(s), \qquad (2.3)
\end{aligned}$$

for $Re(s)>1$. Note that since the equality holds for all s with $Re(s)>1$, it must hold everywhere.

On the other hand, we would like to consider the integral $\int_{C_n} F(z)z^{s-1}\frac{dz}{z}$ where $C_n := (-(2n+1)\pi, -\epsilon) + \gamma^+ + (-\epsilon, -(2n+1)\pi) + \gamma_n$ with $\gamma_n(t) := \{(2n+1)e^{-it} : t \in [-\pi, \pi]\}$ and γ^+ defined as above. Since $F(z)z^{s-2}$ has simple poles at $z = 2\pi i m$ where m is an integer, by the residue theorem, one has

$$\int_{C_n} F(z)z^{s-1}\frac{dz}{z} = -2\pi i \sum_{\substack{m=-n \\ m\neq 0}}^{n} Res_{z=2\pi i m}\left(F(z)z^{s-2}\right).$$

For positive m, one can see that

$$Res_{z=2\pi i m}\left(F(z)z^{s-2}\right) = \lim_{z\to 2\pi i m}(z-2\pi i m)\frac{z^{s-1}e^z}{e^z-1} = -i(2\pi m)^{s-1}e^{s\pi i/2},$$

and for $-m$ (with $m > 0$),

$$Res_{z=-2\pi i m}\left(F(z)z^{s-2}\right) = i(2\pi m)^{s-1}e^{-s\pi i/2}.$$

Hence we have

$$\begin{aligned} \int_{C_n} F(z)z^{s-1}\frac{dz}{z} &= -2\pi(2\pi)^{s-1}\left(e^{\pi i s/2} - e^{-\pi i s/2}\right)\sum_{m=1}^{n} m^{s-1} \\ &= -2^{s+1}\pi^s i \sin\left(\frac{\pi s}{2}\right)\sum_{m=1}^{n} m^{s-1} \end{aligned}$$

If we take $n \to \infty$, the right hand side gives $-2^{s+1}\pi^s i\sin(\pi s/2)\zeta(1-s)$, and we claim that the left hand side approaches $H(s)$ if $Re(s) < 0$. This is because $|F(z)z^{s-1}| = |z^s/(1-e^{-z})| \to 0$ as $n \to \infty$ for $Re(s) < 0$. So,

$$H(s) = -2^{s+1}\pi^s i\sin(\pi s/2)\zeta(1-s) \tag{2.4}$$

for all s with $Re(s) < 0$, and hence it is true for all s.
By (2.3) and (2.4), we have

$$\sin(\pi s)\zeta(s)\Gamma(s) = 2^s\pi^s\sin(\pi s/2)\zeta(1-s),$$

or equivalently,

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos(\pi s/2)\zeta(s)\Gamma(s).$$

Now, We are going to use some properties of the gamma function. The proofs or those properties are omitted here, but can be found, for example, in [1], [2], etc. Applying the above equation as well as $\Gamma(s)\Gamma(1-s) = \pi/(\sin(\pi s))$, which also provides, by substituting $s \mapsto (1-s)/2$, $\Gamma((1+s)/2)\Gamma((1-s)/2) = \pi/\cos(\pi s/2)$, and the duplication formula $\Gamma(s/2) = \pi^{1/2}2^{1-s}\Gamma(s)\Gamma((s+1)/2)^{-1}$, we will complete the proof:

$$\begin{aligned} \pi^{(s-1)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) &= \pi^{-(s+1)/2}2^{1-s}\cos\left(\frac{\pi s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\Gamma(s)\zeta(s) \\ &= \pi^{(1-s)/2}2^{1-s}\Gamma(s)\zeta(s)\Gamma\left(\frac{1+s}{2}\right)^{-1} \\ &= \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s), \end{aligned}$$

□

We introduce the *k-th Bernoulli number*, B_k, defined by the formula

$$F(t) = \frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Notice that

$$\begin{aligned} \sum_{k=0}^{\infty} (-1)^k B_k \frac{t^k}{k!} &= \frac{-t}{e^{-t} - 1} = \frac{te^t - t}{e^t - 1} + \frac{t}{e^t - 1} \\ &= t + \sum B_k \frac{t^k}{k!}, \end{aligned}$$

and it follows that $B_1 = -1/2$ and $B_k = 0$ for all the odd integer k greater than 1.
I also better mention that there is an alternative definition of the Bernoulli numbers, which says $F(t) = te^t/(e^t - 1)$. Note that, with this definition, only the sign of B_1 differs from our original definition because $te^t/(e^t - 1) = t/(e^t - 1) + t$. The second definition seems to fit better for some parts of the theory, as one can see in the following theorem. However, I would like to follow what Washington calls "becoming standard usage" in [8].

Theorem 2.5. $\zeta(0) = -\frac{1}{2}$, *and for any positive integer* $k \geq 2$, $\zeta(1-k) = -\frac{B_k}{k}$.

Proof. Let $F(z)$ and $H(s)$ be same as the ones defined in the proof of the prevous proposition. Notice that $F(z) = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} + t$, and so the residue of $F(z)z^{-k-1}$ at $z = 0$ is $B_k/k!$ if $k > 1$, and $B_1 + 1$ if $k = 1$. Therefore, we only need to show that the residue of $F(z)z^{-k-1}$ at $z = 0$ is $-\zeta(1-k)/(k-1)!$. Recall that

$$H(1-k) = \int_{|z|=\epsilon} F(z)z^{-k-1}dz = 2\pi i Res_{z=0} F(z)z^{-k-1}. \tag{2.6}$$

On the other hand, (2.3) and $\Gamma(s)\Gamma(1-s) = \pi/(\sin(\pi s))$ show that

$$H(s) = -2\pi i \zeta(s)\Gamma(1-s)^{-1}.$$

Let $s = 1-k$ and apply $\Gamma(k) = (k-1)!$ (this is true for$k$ such that $k \in \mathrm{Z}$ and $k > 0$), then we obtain

$$H(1-k) = -2\pi i \zeta(1-k)\Gamma(k)^{-1} = -2\pi i \zeta(1-k)(k-1)!. \tag{2.7}$$

The proof is completed by (2.6) and (2.7). $\square$

Theorem 2.8. *For any positive integer* k, $\zeta(2k) = (-1)^{k-1} \frac{(2\pi)^{2k}}{2 \cdot (2k)!} B_{2k}$.

Proof. The functional equation for $\zeta(s)$, $\zeta(s) = \pi^{s-1/2}\Gamma((1-s)/2)\Gamma(s/2)^{-1}\zeta(1-s)$ (see (2.2)), and the duplication formula for the gamma function, $\Gamma(s/2) = \pi^{1/2}2^{1-s}\Gamma(s)\Gamma((s+1)/2)^{-1}$ (note $\Gamma((s+1)/2)^{-1}$ is well-defined for s positive integer, which is what we consider) give that

$$\zeta(s) = \pi^s 2^{s-1}\Gamma(s)^{-1}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Apply $\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)=\pi\left(\sin\frac{\pi(1-s)}{2}\right)^{-1}$, and we get

$$\zeta(s)=\pi^s 2^{s-1}\Gamma(s)^{-1}\left(\sin\frac{\pi(1-s)}{2}\right)^{-1}\zeta(1-s).$$

By letting $s=2k$ and using Theorem 2.3, the desired equality is obtained. □

3. p-adic Distributions

The purpose of this section is to introduce some important definitions and theorems that we will need to construct the p-adic zeta function.

Let p be any prime. For any nonzero integer x, we define the p-adic ordinal of x, denoted as $\mathrm{ord}_p x$, to be the highest power of p that divides x. In case $x=0$, let $\mathrm{ord}_p x=\infty$. Further, for any rational number x/y, we define $\mathrm{ord}_p\,(x/y)=\mathrm{ord}_p x-\mathrm{ord}_p y$. Now, we can define a map, $|\quad|_p$ on Q as,

$$|x|_p:=\begin{cases}\frac{1}{p^{\mathrm{ord}_p x}} & \text{if } x\neq 0\\ 0 & \text{if } x=0.\end{cases}$$

Proposition 3.1. $|\quad|_p$ *is a non-archimedean norm on* Q.

Proof. $|\quad|_p$ clearly maps all the values x to non-negative real numbers, and $|x|_p=0$ if and only if $x=0$ by the definition. Also, notice that $\mathrm{ord}_p(xy)=\mathrm{ord}_p x+\mathrm{ord}_p y$, and so it follows that $|xy|_p=|x|_p|y|_p$.

Finally, we need to show that $|x+y|_p\leq\max\{|x|_p,\,|y|_p\}$. If one of x, y, or $x+y$ is 0, this is obvious, so we may assume that none of them are 0. Write $x=a/b$ and $y=c/d$. Then, we have

$$\begin{aligned}\mathrm{ord}_p(x+y)&=\mathrm{ord}_p(ad+bc)-\mathrm{ord}_p(bd)\\&\geq\min\{\mathrm{ord}_p(ad),\,\mathrm{ord}_p(bc)\}-\mathrm{ord}_p(bd)\\&=\min\{\mathrm{ord}_p(a/b),\,\mathrm{ord}_p(c/d)\}.\end{aligned}$$

Therefore

$$|x+y|_p=\frac{1}{p^{\mathrm{ord}_p(x+y)}}\leq\max\left\{\frac{1}{p^{\mathrm{ord}_p x}},\,\frac{1}{p^{\mathrm{ord}_p y}}\right\}\leq\max\{|x|_p,\,|y|_p\},$$

and this completes the proof.

□

Let $\{a_i\}$ and $\{b_i\}$ be Cauchy sequences of rational numbers. They are said to be equivalent if $|a_i-b_i|_p\to 0$ as $i\to\infty$. The set, Q_p, of such equivalence classes is what we call the field of *p-adic numbers*. We may also express Q_p as $\mathrm{Q}_p=\left\{\sum_{i=-m}^{\infty}\alpha_i p^i:\alpha_i\in\{0,1,\cdots,p-1\}\right\}$. We now extend the p-adic norm to Q_p by defining $|a|_p:=\lim_{i\to\infty}|a_i|$ where $\{a_i\}$ is a representative of an equivalence class a.

The set of *p-adic integers*, Z_p, and the set of *p-adic units*, $\mathrm{Z}_p^\times$, are defined as follows: $\mathrm{Z}_p:=\{a\in\mathrm{Q}_p:|a|_p\leq 1\}$, and $\mathrm{Z}_p^\times:=\{a\in\mathrm{Q}_p:|a|_p=1\}$. Note that, in terms of p-adic expansion, it means that

$$\mathrm{Z}_p = \left\{ \sum_{i=0}^{\infty} a_i p^i : a_i \in \{0, 1, \cdots, p-1\} \right\} \text{ and } \mathrm{Z}_p^{\times} = \left\{ \sum_{i=0}^{\infty} a_i p^i : a_i \in \{0, 1, \cdots, p-1\}, a_0 \neq 0 \right\}.$$

Finally, for fixed $a \in \mathrm{Q}_p$ and $N \in \mathrm{Z}$, we define an *interval*, $a + p^N \mathrm{Z}_p$, which is the set of elements in Q_p whose distance from a is less than p^{-N}, i.e., $a + p^N \mathrm{Z}_p := \{x \in \mathrm{Q}_p : |x - a|_p \leq 1/p^N\}$. The intervals form a basis of open sets on Q_p. A simpler notation, $a + (p^N)$, is sometimes substituted for $a + p^N \mathrm{Z}_p$. We claim that any compact-open subset of Q_p can be expressed as a finite disjoint union of such intervals. For the proof, let us take any compact-open subset, X, of Q_p. Since X is open, for each $a \in X$, one can find an interval $a + \left(p^N\right)$ with N large enough so that $a + \left(p^N\right) \subset X$. Hence $X = \bigcup_{a \in X} \left(a + \left(p^{N_a}\right)\right)$ is an open cover of X, and the compactness of X gives a finite subcover, say $\bigcup_{i=1}^{m} \left(a_i + \left(P^{N_i}\right)\right)$. For the disjointness, choose the largest N_i and write

$$a_j + \left(P^{N_j}\right) = \bigcup_{b=1}^{P^{N_i - N_j} - 1} \left(a_j + bP^{N_j} + \left(P^{N_i}\right)\right)$$

for each j. This completes the proof.

Definition 3.2. Let X be a compact-open subset of Q_p. A *p-adic distribution* μ *on* X is an additive map from the set of compact-open sets in X to Q_p, i.e., if U is compact-open in X and is a finite disjoint union of compact-open subsets, $\{U_i\}_{i=1}^n$, then

$$\mu\left(U\right) = \sum_{i=1}^{n} \mu\left(U_i\right).$$

A p-adic distribution μ on X is called a *measure* if there exists a positive real number, M, such that $|\mu(U)|_p \leq M$ for all compact-open sets U in X.

Proposition 3.3. *Let μ be a map from the set of compact-open in X to Q_p such that*

$$\mu\left(a + (p^N)\right) = \sum_{b=0}^{p-1} \mu\left(a + bp^N + (p^{N+1})\right) \quad \textit{for any interval } a + (p^N) \textit{ in } X.$$

Then μ extends uniquely to a p-adic distribution on X.

Proof. Let U be a compact-open subset of X. We already proved that U can be written as a finite disjoint union of intervals, say $U = \bigcup_{i=1}^{n} I_i$ with $I_i = a_i + (p^N)$ for some a_i and N. Define $\mu(U) := \sum \mu(I_i)$.

First we will prove that it is well-defined: Suppose that $U = \bigcup I_i = \bigcup I_i'$, where $\{I_i\}$ and $\{I_i'\}$ are different partitions. If $I_i \bigcap I_j' \neq \phi$, let $I_{ij} := I_i \bigcap I_j'$. Then $I_i = \bigcup_j I_{ij}$ and there exists $N' > N$ such that $I_{ij} = a_i + \sum_{k=N}^{N'-1} a_{jk} p^k + (p^{N'})$ for each j. By the hypothesis, we have $\mu(I_i) = \sum_j \mu(I_{ij})$, and so

$$\mu(U) = \sum_{i} \mu(I_i) = \sum_{i,j} \mu(I_{ij}).$$

Similarly, $\mu(U) = \sum_j \mu(I_j') = \sum_{i,j} \mu(I_{ij})$, thus μ is independent of the choice of the partitions.

To show that μ is an additive map, let U be compact-open in X such that $U = \bigcup_{i=1}^{n} U_i$, where $\{U_i\}$ are disjoint compact-open subsets. We know that each U_i can be written as a finite disjoint union of intervals, say I_{ij}. But then, it immediately follows from our definition of μ that

$$\mu(U) = \mu\left(\bigcup I_{ij}\right) = \sum_{i,j} \mu(I_{ij}) = \sum_i \sum_j \mu(I_{ij}) = \sum_i \mu(U_i).$$

□

In order to introduce a p-adic distribution, we first would like to define the *Bernoulli polynomials*. The k-th Bernoulli polynomial is defined as

$$F(x,t) = F(t)e^{xt} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!},$$

where $F(t) = t/(e^t - 1)$, as it is defined in the previous section. The first few Bernoulli polynomials are given as:

$$B_0(x) = 1, \qquad B_1(x) = x - \frac{1}{2}, \qquad B_2(x) = x^2 - x + \frac{1}{6}.$$

In general, one can show that $B_k(x) = \sum_{i=0}^{k} \binom{k}{i} B_i x^{k-i}$, where B_k is the k-th Bernoulli number. In particular, $B_k(0) = B_k$ for all k. To prove this property, consider:

$$\begin{aligned} \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!} &= \frac{t}{e^t-1}e^{xt} = \left(\sum_{k=0}^{\infty} B_k \frac{t^k}{k!}\right)\left(\sum_{k=0}^{\infty} \frac{x^k t^k}{k!}\right) \\ &= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} \frac{B_i}{i!}\frac{x^{k-i}}{(k-i)!}\right) t^k = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\binom{k}{i} B_i\right)\frac{t^k}{k!}. \end{aligned}$$

Using the k-th Bernoulli polynomial defined above, let us define a map μ_k to be

$$\mu_k\left(a + \left(p^N\right)\right) := p^{N(k-1)} B_k\left(\frac{a}{p^N}\right) \quad \text{for some} \quad a \in \{0,1,2,\cdots,p^N - 1\}.$$

Proposition 3.4. *μ_k extends to a distribution on* Z_p.

Proof. Since Z_p is a compact-open set in Q_p, it is enough to show, by Proposition 3.3, that $\mu_k(a+\left(p^N\right)) = \sum_{b=0}^{p-1} \mu_k\left(a + bp^N + \left(p^{N+1}\right)\right)$, i.e. $B_k\left(\frac{a}{p^N}\right) = p^{k-1}\sum_{b=0}^{p-1} B_k\left(\frac{a}{p^{N+1}} + \frac{b}{p}\right)$. Let $\alpha = a/p^N$, then

$$\begin{aligned} \sum_{b=0}^{p-1}\sum_{k=0}^{\infty} B_k\left(\alpha + \frac{b}{p}\right)\frac{t^k}{k!} &= \sum_{b=0}^{p-1}\frac{te^{(\alpha+b/p)t}}{e^t-1} = \frac{te^{\alpha t}}{e^t-1}\sum_{b=0}^{p-1} e^{(t/p)b} = \frac{te^{\alpha t}}{e^t-1}\cdot\frac{e^{(t/p)\cdot p}-1}{e^{t/p}-1} \\ &= p\cdot\frac{(t/p)e^{\alpha t}}{e^{t/p}-1} = p\sum_{k=0}^{\infty}\frac{B_k(\alpha)}{p^k}\frac{t^k}{k!}. \end{aligned}$$

This completes the proof. □

This distribution, μ_k, is called the *k-th Bernoulli distribution.* Note that μ_k is not a measure for any nonnegative integer k. However, there is a method to *regularize* the Bernoulli distribution to obtain a measure.

Definition 3.5. Let α be any rational integer, not equal to one and that p does not divide α. We define the *k-th regularized Bernoulli distribution on* Z_p as

$$\mu_{k,\alpha}(U) = \mu_k(U) - \alpha^{-k}\mu_k(\alpha U),$$

where $\alpha U := \{x \in \mathrm{Q}_p : x/\alpha \in U\}$.

We must show that $\mu_{k,\alpha}$ is well-defined: It is clear that the sum of two distributions is a distribution and that for any $\alpha \in \mathrm{Z}_p$ and a distribution μ on Z_p, $\alpha\mu$ is a distribution as well. Thererfore, we only need to show that $\mu(\alpha U)$ is a distribution. It can be shown as follows: Write U as a finite disjoit union of intervals, say $\{U_i\}_{i=1}^n$. Then $x \in \alpha U \Leftrightarrow x/\alpha \in U \Leftrightarrow x/\alpha \in U_i$ for a unique $i \Leftrightarrow x \in \alpha U_i$ for a unique i. This proves that αU is a disjoint union of $\{aU_i\}_{i=1}^n$, and that $\mu(\alpha U) = \sum \mu(\alpha U_i)$.

For any α in Z_p, let $\{\alpha\}_N$ be a unique rational integer such that $0 \le \{\alpha\}_N \le p^N - 1$ and $\{\alpha\}_N \equiv \alpha \pmod{p^N}$. If $U = a + (p^N)$ for some $a \in \{0, 1, \cdots, p^N - 1\}$, then

$$\begin{aligned} \alpha U &= \{x \in \mathrm{Z}_p : x/\alpha \in U\} = \left\{x \in \mathrm{Z}_p : |x/\alpha - a|_p \le\ p^{-N}\right\} \\ &= \left\{x \in \mathrm{Z}_p : |1/\alpha|_p\,|x - \alpha a|_p \le p^{-N}\right\} \\ &= \{\alpha a\}_N + (p^N). \end{aligned}$$

Now, we are ready to take a closer look on $\mu_{k,\alpha}$ for each k.
First, let $k = 0$. Then $\mu_{0,\alpha}\left(a + \left(p^N\right)\right) = \mu_0\left(a + \left(p^N\right)\right) - \mu_0\left(\{\alpha a\}_N + \left(p^N\right)\right) = p^{-N} - p^{-N} = 0$, for any interval, $a + \left(p^N\right)$. This result does not lead us anywhere, therefore let us consider the case $k \ge 1$.

Proposition 3.6. *$\mu_{k,\alpha}$ is a measure for any rational integer, α, not divisible by p and for all $k \ge 1$.*

Proof. First, we would like to consider the case $k = 1$;

$$\begin{aligned} \mu_{1,\alpha}\left(a + \left(p^N\right)\right) &= \mu_1\left(a + \left(p^N\right)\right) - \alpha^{-1}\mu_1\left(\{\alpha a\}_N + \left(p^N\right)\right) \\ &= B_1\left(\frac{a}{p^N}\right) - \alpha^{-1}B_1\left(\frac{\{\alpha a\}_N}{p^N}\right) \\ &= \frac{a}{p^N} - \frac{1}{2} - \frac{1}{\alpha}\left(\frac{\{\alpha a\}_N}{p^N} - \frac{1}{2}\right) \\ &= \frac{1}{\alpha}\left[\frac{\alpha a}{p^N}\right] + \frac{1}{2}\left(\frac{1}{\alpha} - 1\right). \end{aligned}$$

We claim that this is a measure. It is enough to show that $\left|\mu_{1,\alpha}\left(a + \left(p^N\right)\right)\right|_p$ is bounded for any $a \in \{0, 1, \cdots, p^N - 1\}$ because any compact-open subset of Z_p is a finite disjoint union of intervals. Note that α is not divisible by p, i.e. α is in $\mathrm{Z}_p^\times$, therefore $1/\alpha \in \mathrm{Z}_p^\times$ as well. So if $p \ne 2$, then $1/2\,(1/\alpha - 1) \in \mathrm{Z}_p$. If $p = 2$, we may write $1/\alpha = 1 + \sum_{i=0}^{\infty} a_i 2^i$ with $a_i \in \{0, 1\}$ since $1/\alpha \in \mathrm{Z}_p^\times$. Hence $1/2\,(1/\alpha - 1)$ is, again, in Z_p.

Since $\dfrac{1}{\alpha}\left[\dfrac{\alpha a}{p^N}\right]$ is also in Z_p, it follows that $\mu_{1,\alpha}\left(a+\left(p^N\right)\right)\in \mathrm{Z}_p$ for any interval $a+\left(p^N\right)$. It follows that $\mu_{1,\alpha}$ is a measure. More precisely, $|\mu_{1,\alpha}(U)|_p$ is bounded by 1.

To complete the proof, we need the following lemma.

Lemma 3.7. *Let d_k be the least common multiple of denominators of the coefficients of $B_k(x)$. Then*

$$\mu_{k,\alpha}\left(a+\left(p^N\right)\right)\equiv ka^{k-1}\mu_{1,\alpha}\left(a+\left(p^N\right)\right)\quad (\mathrm{mod}\ p^{N-ord_p d_k}).$$

Proof. The equivalence in Lemma can be rephrased as

$$d_k\mu_{k,\alpha}\left(a+\left(p^N\right)\right)\equiv d_k k a^{k-1}\mu_{1,\alpha}\left(a+\left(p^N\right)\right)\quad (\mathrm{mod}\ p^N),$$

and we wish to show this equivalence. By the definition of $\mu_{k,\alpha}$, we have

$$\begin{aligned} d_k\cdot\mu_{k,\alpha}(a+\left(p^N\right)) &= d_k\left(\mu_k(a+\left(p^N\right))-\alpha^{-k}\mu_k(\{a\alpha\}_N+\left(p^N\right))\right)\\ &= d_k p^{N(k-1)}B_k\left(\frac{a}{p^N}\right)-d_k\alpha^{-k}p^{N(k-1)}B_k\left(\frac{\{a\alpha\}_N}{p^N}\right). \end{aligned}\tag{3.8}$$

Now, recall that $B_k(x)=\sum_{i=0}^{k}\binom{k}{i}B_i x^{k-i}=x^k-(k/2)x^{k-1}+\cdots+B_k$, since $B_0=1$ and $B_1=-1/2$. Hence, the first part of the right hand side of (2.8) is:

$$\begin{aligned} d_k p^{N(k-1)}B_k\left(\frac{a}{p^N}\right) &\equiv d_k p^{N(k-1)}\left(\frac{a^k}{p^{Nk}}-\frac{k}{2}\frac{a^{k-1}}{p^{N(k-1)}}\right)\quad (\mathrm{mod}\ p^N)\\ &= d_k a^{k-1}\left(\frac{a}{p^N}-\frac{k}{2}\right). \end{aligned}$$

Also, the second part of (3.8) is:

$$\begin{aligned} & d_k\alpha^{-k}p^{N(k-1)}B_k\left(\frac{\{a\alpha\}_N}{p^N}\right)\\ \equiv\ & d_k\alpha^{-k}p^{N(k-1)}\left(\frac{\{a\alpha\}_N^k}{p^{Nk}}-\frac{k}{2}\frac{\{a\alpha\}_N^{k-1}}{p^{N(k-1)}}\right)\quad (\mathrm{mod}\ p^N)\\ =\ & d_k\alpha^{-k}p^{N(k-1)}\left(\left(\frac{a\alpha}{p^N}-\left[\frac{a\alpha}{p^N}\right]\right)^k-\frac{k}{2}\left(\frac{a\alpha}{p^N}-\left[\frac{a\alpha}{p^N}\right]\right)^{k-1}\right)\\ \equiv\ & d_k\alpha^{-k}p^{N(k-1)}\left(\frac{a^k\alpha^k}{p^{Nk}}-k\frac{a^{k-1}\alpha^{k-1}}{p^{N(k-1)}}\left[\frac{a\alpha}{p^N}\right]-\frac{k}{2}\left(\frac{a^{k-1}\alpha^{k-1}}{p^{N(k-1)}}\right)\right)\quad (\mathrm{mod}\ p^N)\\ =\ & d_k a^{k-1}\left(\frac{a}{p^N}-\frac{k}{\alpha}\left[\frac{a\alpha}{p^N}\right]-\frac{k}{2\alpha}\right). \end{aligned}$$

Simplify them, and we obtain

$$\begin{aligned} d_k\mu_{k,\alpha}(a+\left(p^N\right)) &\equiv d_k k a^{k-1}\left(\frac{1}{\alpha}\left[\frac{a\alpha}{p^N}\right]+\frac{k}{2}\left(\frac{1}{\alpha}-1\right)\right)\quad (\mathrm{mod}\ p^N)\\ &= d_k k a^{k-1}\mu_{1,\alpha}(a+\left(p^N\right)), \end{aligned}$$

and the proof is completed. □

This lemma shows that $\left|\mu_{k,\alpha}\left(a+\left(p^N\right)\right)\right|_p$ is bounded by $\max\left(\left|\mu_{1,\alpha}\left(a+\left(p^N\right)\right)\right|_p, \left|p^N/d_k\right|_p\right)$, and so it completes the proof of the proposition above.

Definition 3.9. Let μ be a p-adic measure on a compact-open set X in $\mathbb{Q}_p$, and $f : X \to \mathbb{Q}_p$ a continuous function. Then the *N-th Riemann sum* is defined as

$$S_{N,\{x_{a_i,N}\}}(f) = \sum_{i=1}^{m} f(x_{a_i,N})\mu\left(a_i+\left(p^N\right)\right)$$

where X is expressed as a disjoint union of $\{a_i+\left(p^N\right)\}_{i=1}^m$, and $x_{a_i,N}$ are arbitrary points in $a_i+\left(p^N\right)$ for each i.

Theorem 3.10. $\lim_{N\to\infty} S_{N,\{x_{a_i,N}\}}$ *exists in* $\mathbb{Q}_p$*, and it is independent of the choice of* $\{x_{a_i,N}\}$.

Proof. There exists $M>0$ such that $|\mu(U)|_p \leq M$ for any compact-open subset U of X. Let $\epsilon>0$ and choose N large enough so that it satisfies the following two conditions:

(1) X can be written as a finite disjoint union of intervals, $\left\{a_i+\left(p^N\right)\right\}_{i=1}^n$, where a_i is nonnegative integer between 0 and p^N-1 for all i.

(2) For any x and y with $|x-y|_p \leq p^{-N}$, $|f(x)-f(y)|_p < \epsilon/M$. (Such N exists since the compactness of X guarantees the uniform continuity of f.)

If $N' > N$, then $X = \bigcup_{i=1}^n \left(a_i+\left(p^N\right)\right)$ has subpartitions, say $X = \bigcup_{i,j}\left(a_{ij}+\left(p^{N'}\right)\right)$. Hence, we have

$$\begin{aligned} S_{n,\{x_{a_i,N}\}} &= \sum_i f(x_{a_i,N})\mu\left(a_i\left(p^N\right)\right) \\ &= \sum_i f(x_{a_i,N})\sum_j \mu\left(a_{ij}+\left(p^{N'}\right)\right) \\ &= \sum_{i,j} f(x_{a_i,N})\mu\left(a_{ij}+\left(p^{N'}\right)\right), \end{aligned}$$

and

$$\begin{aligned} \left|S_{N,\{x_{a_i,N}\}} - S_{N',\{x_{a_{ij},N'}\}}\right|_p &= \left|\sum_{i,j}\left(f(x_{a_i,N})-f(x_{a_{ij},N'})\right)\mu\left(a_{ij}+\left(p^{N'}\right)\right)\right|_p \\ &\leq \max_{i,j}\left|f(x_{a_i,N})-f(x_{a_{ij},N'})\right|_p \left|\mu\left(a_{ij}+\left(p^{N'}\right)\right)\right|_p \leq \epsilon. \end{aligned}$$

Thus, the limit exists, and the limit must be in $\mathbb{Q}_p$ by the completeness of $\mathbb{Q}_p$.

To show the independence of the choice of points, choose any point $y_{a_i,N}$ from each interval $a_i+\left(p^N\right)$. Similar to the argument above, one can show that $|S_{N,\{x_{a_i,N}\}} - S_{N,\{y_{a_i,N}\}}|_p \leq \epsilon$. □

We define $\displaystyle\int f\mu$ to be this limit in the above theorem. Note that it is well-defined by the theorem.

Proposition 3.11. *If $f, g : X \to \mathbb{Q}_p$ are continuous functions such that $|f(x) - g(x)|_p \leq \epsilon$ for all x, and $|\mu(U)|_p \leq M$ for all compact-open U, then*

$$\left| \int f\mu - \int g\mu \right|_p \leq M\epsilon.$$

The proof of this proposition is trivial, for $\left|\int f\mu - \int g\mu\right|_p \leq \int |f-g|_p |\mu|_p \leq M\epsilon$.

4. The p-adic Zeta Function

We wish to have the continuity of $f(s) = n^s$ where s is in $\mathbb{Z}_p$. However, it depends on n. For example, take s and s' very close p-adically with $s < s'$. But if n is in $p\mathbb{Z}_p$ then $|n^s - n^{s'}|_p = |n^s|_p|1 - n^{s'-s}|_p = |n^s|_p$, and so n^s is not continuous. So now, we need to restrict n to be a p-adic unit. We first show that $|n^s - n^{s'}|_p$ converges to zero if s and s' are congruent modulo $p-1$. Fix s_0 to be an element in $\{1, 2, \cdots, p-2\}$, and define $\mathcal{A}_{s_0} := \{s \in \mathbb{Z}, s > 0 : s \equiv s_0 \pmod{p-1}\}$. Then for any s in $\mathcal{A}_{s_0}$, we have $|n^{s_0} - n^s|_p = |n^{s_0}|_p|1 - n^{s-s_0}|_p = |n^{s_0}|_p|1 - n^{(p-1)t}|_p$ for some integer t. But the exact sequence, $1 \to 1 + p\mathbb{Z}_p \to \mathbb{Z}_p^\times \to \mathbb{F}_p^* \to 1$, shows that $n^{p-1} \equiv 1 \pmod{p}$. Hence, we may write $n^{p-1} = 1 + mp$ for some m, and if s and s_0 are very close, say $s - s_0 = (p-1)p^N s'$ for some rational integer s', then we have

$$\begin{aligned} |n^{s_0} - n^s|_p &= |n^{s_0}|_p \left|1 - n^{(p-1)p^N s'}\right|_p \\ &= \left| -\sum_{k=1}^{p^N s'} \binom{p^N s'}{k} (mp)^k \right|_p \leq |p^{N+1}|_p = \frac{1}{p^{N+1}}. \end{aligned}$$

Thus, one can have the continuity of $f(s) = n^s$ for this case.
Now, we would like to extend our definition to $\mathbb{Z}_p$. But this follows from the claim that $\mathcal{A}_{s_0}$ is dense in $\mathbb{Z}_p$. To prove this claim, it is enough to show that, for any s in $\mathbb{Z}_p$, there exists a sequence $\{s_i\}$ in $\mathcal{A}_{s_0}$ that converges to s. Let us write $s = \sum_{i=0}^\infty a_i p^i$ and let $s_i = \sum_{j=0}^i a_j p^j + (s_0 - a_0 - a_1 - \cdots - a_i)p^i$. We claim that such s_i are in $\mathcal{A}_{s_0}$ for all i. This is because $p^j \equiv 1 \pmod{p-1}$ for any j, which follows from $p^j - 1 = (p-1)(p^{j-1} + \cdots p + 1) \equiv 0 \pmod{p-1}$, and so $s_i \equiv a_0 + a_1 + \cdots + a_i + s_0 - a_0 - \cdots - a_i = s_0 \pmod{p-1}$.
Clearly $\{s_i\}$ converges to s, for $|s_i - s|_p = \left|(s_0 - a_0 - \cdots - a_i)p^i - \sum_{j=i+1}^\infty a_j p^j\right|_p \leq 1/p^i \to 0$ as $i \to \infty$. Thus, the claim is proven and we can extend a continuous function $f(s) = n^s$ to $\mathbb{Z}_p$.

The above argument and Proposition 3.11 say that if x is a p-adic unit and $k \equiv k' \pmod{(p-1)p^N}$, then $|x^{k-1} - x^{k'-1}|_p \leq 1/p^{N+1}$, and so we also obtain

$$\left| \int_{\mathbb{Z}_p^\times} x^{k-1} \mu_{1,\alpha} - \int_{\mathbb{Z}_p^\times} x^{k'-1} \mu_{1,\alpha} \right|_p \leq \frac{1}{p^{N+1}}.$$

This result will be used in some of the proofs for the rest of this section.

Definition 4.1. Let α be a rational integer that is not equal to one and not divisible by p. For any positive integer k, we define

$$\zeta_p(1-k) = \frac{1}{\alpha^{-k}-1}\int_{\mathrm{Z}_p^\times} x^{k-1}\mu_{1,\alpha}.$$

One can show that this is well-defined by using the following lemma.

Lemma 4.2. *Let k be a positive integer, and let X be compact-open in Z_p. Then*

$$\int_X \mu_{k,\alpha} = k\int_X x^{k-1}\mu_{1,\alpha}.$$

Proof. Write X as a finite disjoint union of intervals, say $X = \bigcup_{i=1}^n (a_i + (p^N))$ for N large enough. Then

$$\int_X 1\mu_{k,\alpha} = \sum_{i=1}^n \int_{a_i+(p^N)} 1\mu_{k,\alpha} = \sum_{i=1}^n \mu_{k,\alpha}(a_i + (p^N)).$$

But, since $\mu_{k,\alpha}\left(a+\left(p^N\right)\right) \equiv ka^{k-1}\mu_{1,\alpha}\left(a+\left(p^N\right)\right) \pmod{p^{N-\mathrm{ord}_p d_k}}$ by Lemma 3.7, we have

$$\begin{aligned}\int_X 1\mu_{k,\alpha} &\equiv k\sum_{i=1}^n a_i^{k-1}\mu_{1,\alpha}\left(a_i+(p^N)\right) \pmod{p^{N-\mathrm{ord}_p d_k}}\\ &= k\sum_{i=1}^n f(a_i)\mu_{1,\alpha}\left(a_i+(p^N)\right) = k\cdot S_{N,\{x_{a_i,N}\}}(f).\end{aligned}$$

Therefore, by taking $N\to\infty$, we obtain the desired result. $\square$

This shows that:

$$\begin{aligned}\frac{1}{\alpha^{-k}-1}\int_{\mathrm{Z}_p^\times} x^{k-1}\mu_{1,\alpha} &= \frac{1}{\alpha^{-k}-1}\int_{\mathrm{Z}_p^\times} 1\mu_{k,\alpha}\\ &= \frac{1}{\alpha^{-k}-1}\frac{1}{k}\mu_{k,\alpha}(\mathrm{Z}_p^\times)\\ &= \frac{1}{(\alpha^{-k}-1)k}\left(\mu_k(\mathrm{Z}_p^\times) - \alpha^{-k}\mu_k(\mathrm{Z}_p^\times)\right)\\ &= -\frac{\mu_k(\mathrm{Z}_p^\times)}{k},\end{aligned}$$

and the right hand side of the definition 4.1 does not depend on α. So it is well-defined, and moreover, it has a following property.

Proposition 4.3. $\zeta_p(1-k) = \left(1-p^{k-1}\right)\left(-\dfrac{B_k}{k}\right)$, *where B_k is the k-th Bernoulli number.*

Proof. By the definition of ζ_p and the lemma above, we have

$$\zeta_p(1-k) = \frac{1}{k\left(\alpha^{-k}-1\right)}\int_{\mathrm{Z}_p^\times} 1\mu_{k,\alpha} = \frac{1}{k\left(\alpha^{-k}-1\right)}\mu_{k,\alpha}(\mathrm{Z}_p^\times).$$

We claim that $\mu_{k,\alpha}(\mathrm{Z}_p^\times) = \left(1-\alpha^{-k}\right)\left(1-p^{k-1}\right)B_k$. This can be easily shown by using the definitions: Since $\mu_k(\mathrm{Z}_p) = p^0 B_k(0) = B_k$ and $\mu_k(p\mathrm{Z}_p) = p^{k-1}B_k(0) = p^{k-1}B_k$, $\mu_k(\mathrm{Z}_p^\times) =$

$\mu_k(\mathrm{Z}_p) - \mu_k(p\mathrm{Z}_p) = \left(1 - p^{k-1}\right) B_k$. Also note that, for any p-adic unit α, $\alpha\mathrm{Z}_p^\times = \mathrm{Z}_p^\times$. Thus

$$\mu_{k,\alpha}(\mathrm{Z}_p^\times) = \mu_k(\mathrm{Z}_p^\times) - \alpha^{-k}\mu_k(\alpha\mathrm{Z}_p^\times) = (1-\alpha^{-k})\mu_k(\mathrm{Z}_p^\times) = \left(1-\alpha^{-k}\right)\left(1-p^{k-1}\right) B_k,$$

and this completes the proof of the proposition. □

Note that this proposition says that $\zeta_p(1-k)$ can be obtained by removing the p-factor from the Euler identity for $\zeta(1-k)$ since $B_k/k = -\zeta(1-k)$ by Proposition 2.3.
The next theorem is known as *Kummer congruence*, and it tells us some properties of $\zeta_p(1-k)$.

Theorem 4.4. *(Kummer)*

(1) *If* $(p-1) \nmid k$ *then* $\dfrac{B_k}{k}$ *is a* p*-adic integer.*

(2) *If* $(p-1) \nmid k$ *and* $k \equiv k' \pmod{(p-1)p^N}$ *then*

$$\left(1-p^{k-1}\right)\frac{B_k}{k} \equiv \left(1-p^{k'-1}\right)\frac{B_{k'}}{k'} \pmod{p^{N+1}}.$$

Proof. If $k = 1$, then $|B_k/k|_p = 1$ for any $p > 2$. For the case $k > 1$, choose α such that $2 \le \alpha \le p-1$ and $p-1$ is the smallest positive integer satisfying $\alpha^{p-1} - 1 \equiv 0 \pmod{p}$. Note that such α exists since α is a p-adic unit and it can be identified with a $(p-1)$-th root of unity. By the choice of α and the hypothesis, $\alpha^k - 1 \not\equiv 0 \pmod{p}$, and so $\alpha^{-k} - 1$ is a p-adic unit. Therefore,

$$\left|\frac{B_k}{k}\right|_p = \left|1-p^{k-1}\right|_p^{-1}\left|\alpha^{-k}-1\right|_p^{-1}\left|\int_{\mathrm{Z}_p^\times} x^{k-1}\mu_{1,\alpha}\right|_p \le |\mu_{1,\alpha}(\mathrm{Z}_p^\times)|_p \le 1.$$

To prove the second statement, let α be such that we chose earlier. Since k is congruent to k' $\bmod (p-1)p^N$, we have $\alpha^k \equiv \alpha^{k'} \pmod{p^{N+1}}$. (This follows from the argument in the first part of this section.) Also, By Proposition 3.11, $\int x^{k-1}\mu_{1,\alpha} \equiv \int x^{k'-1}\mu_{1,\alpha} \pmod{p^{N+1}}$. Hence the proof is completed by Proposition 4.2. □

Finally, let us fix $s_0 \in \{0, 1, \cdots, p-2\}$. For any p-adic integer, s, there exists a sequence of positive integer, say $\{t_i\}_{i=1}^\infty$, that converges to s, namely $t_i = \sum_{j=0}^i a_j p^j$ where $s = \sum_{j=0}^\infty a_j p^j$. Therefore, the following limit makes sense:

$$\zeta_{p,s_0}(s) := \lim_{i\to\infty}\left(1-p^{s_0+(p-1)t_i-1}\right)\left(-B_{s_0+(p-1)t_i}/(s_0+(p-1)t_i)\right),$$

unless s and s_0 are both zeros. The p-adic zeta function can be defined in the following way.

Definition 4.5. For any $\alpha \in \mathrm{Z}$ with $\alpha \ne 1$ and $p \nmid \alpha$, and for a fixed integer $s_0 \in \{0, 1, 2, \cdots, p-2\}$, $\zeta_{p,s_0}(s)$ is defined as

$$\zeta_{p,s_0}(s) := \frac{1}{\alpha^{-(s_0+(p-1)s)}-1}\int_{\mathrm{Z}_p^\times} x^{s_0+(p-1)s-1}\mu_{1,\alpha},$$

for any p-adic integer s, except at $s = 0$ in case of $s_0 = 0$.

Note that $\zeta_{p,s_0}(s)$ is continuous except where s_0 and s are both zeros . This can be shown in the same manner as the continuity of $\zeta_p(1-k)$. (See the first paragraph of this section.) One can also show that $\zeta_{p,s_0}(k)$ does not depend on the choice of α for any k in $\mathcal{A}_{s_0}$. This is because $\zeta_p(1-k) = \zeta_{p,s_0}(k_0)$ where $k = s_0 + (p-1)k_0$ for some $s_0 \in \{0, 1, \cdots, p-2\}$ and

$k_0 \in \mathrm{Z}$ with $k_0 > 0$. It follows from the continuity of $\zeta_{p,s_0}(s)$ and the density of $\mathcal{A}_{s_0}$ that $\zeta_{p,s_0}(s)$ is independent of the choice of α.

We also note that $\zeta_p(t)$ has a pole at $t = 1$, by taking $k = 0$ (and so $s_0 = k_0 = 0$ as well) in $\zeta_p(1-k) = \zeta_{p,s_0}(k_0)$.

5. Dirichlet L-functions

In this section, we would like to define the Dirichlet L-functions and observe some of their properties. In order to define the Dirichlet L-function, we first need to know the Dirichlet characters.

Let m be a positive integer. A map $\chi : \mathrm{Z} \to S^1 := \{z \in \mathrm{C} : |z| = 1\}$ is called *Dirichlet character modulo* m if it satisfies the following conditions.

(1) For all a, $b \in \mathrm{Z}$, $\chi(ab) = \chi(a)\chi(b)$
(2) If $a \equiv b \pmod m$ then $\chi(a) = \chi(b)$
(3) $\chi(a) = 0$ if and only if $(a, m) > 1$.

If the character, χ^0, is defined as

$$\chi^0(a) = \begin{cases} 1 & \text{if } (a,m) = 1 \\ 0 & \text{if } (a,m) > 1, \end{cases}$$

then χ^0 is called the *trivial character*. In particular, if $m = 1$, it is called the *principal character* and is denoted as 1. Furthermore, the Dirichlet character χ mod m is called *primitive* if, for any m' that divides m, there does not exist a character χ' such that $\chi(a) = \chi'(a)$ for all a, i.e., m is the smallest integer that defines χ. In this case, m is called the *conductor* of χ, and denoted by f_χ. (We will simply denote f when it is clear from the contexts.)

Remark 1. *A primitive Dirichlet character* χ mod f *satisfies the following conditions:*

(1) $\chi(a) = 1$ *for all* a *such that* $a \equiv 1 \pmod f$.
(2) *If* χ *is nontrivial, then* $\sum_{a=1}^{f} \chi(a) = 0$.

Proof. Choose a such that $(a, f) = 1$. Then, by the third condition of the definition, $\chi(a) \neq 0$. Also, by taking $b = 1$ in the first condition, we may write $\chi(a) = \chi(a)\chi(1)$. Hence $\chi(1) = 1$, and so $\chi(a) = \chi(1) = 1$ for all a such that $a \equiv 1 \pmod f$ by the second condition.

To prove (2), let χ be a nontrivial Dirichlet character, and choose b such that $\chi(b) \neq 0, 1$. (Since χ is nontrivial, such b must exist.) Note that $\sum_{a=1}^{f} \chi(a) = \sum_{a=1}^{f} \chi(ab)$, because for each $a \in \{1, 2, \cdots, f\}$ there exists $a' \in \{1, 2, \cdots, f\}$ such that $a' \equiv ab \pmod f$. Therefore,

$$\chi(b) \sum_{a=1}^{f} \chi(a) = \sum_{a=1}^{f} \chi(ab) = \sum_{a=1}^{f} \chi(a),$$

and we get $(1 - \chi(b)) \sum_{a=1}^{f} \chi(a) = 0$. Since $\chi(b) \neq 1$, the sum must be zero. □

The *Dirichlet L-function associated to* χ is defined by the formula

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

We note that if we take $\chi = 1$, then $L(s,1) = \zeta(s)$.

Proposition 5.1. *$L(s,\chi)$ converges absolutely for $Re(s) > 1$ and satisfies the Euler identity,*

$$L(s,\chi) = \prod_p \frac{1}{1-\chi(p)p^{-s}},$$

where p runs through all the primes.

This can be proved in the exactly same way as the Riemann zeta fucntion (see Proposition 1.1), and the proof is omitted.

The *Gauss sum, $\tau(\chi, n)$, associated to the Dirichlet character χ* is defined as

$$\tau(\chi, n) := \sum_{a=1}^{f} \chi(a) e^{2\pi i a n/f},$$

where f is the conductor of χ. In case of $n = 1$, we simply write $\tau(\chi, 1) = \tau(\chi)$.

Let $\delta = \delta_\chi$ be

$$\delta = \begin{cases} 0 & \text{if } \chi(-1) = 1 \\ 1 & \text{if } \chi(-1) = -1. \end{cases}$$

Proposition 5.2. *Let χ be a nontrivial primitive Dirichlet character* $\bmod f$*, and let*

$$\Lambda(s,\chi) := \left(\frac{f}{\pi}\right)^{s/2} \Gamma\left(\frac{s+\delta}{2}\right) L(s,\chi).$$

Then Λ is called the completed L-series *and satisfies the functional equation*

$$\Lambda(s,\chi) = W(\chi)\Lambda(1-s,\bar{\chi}),$$

where $W(\chi) = \dfrac{\tau(\chi)}{i^\delta \sqrt{f}}$.

Proof. Let $F(z) = \displaystyle\sum_{a=1}^{f} \chi(a) \frac{z e^{az}}{e^{fz}-1}$, $G(z) = F(-z)z^{-1} = \displaystyle\sum_{a=1}^{f} \chi(a) \frac{e^{-az}}{1-e^{-fz}}$, and $H(s) = \displaystyle\int_{C}^{-} F(z) z^{s-1} \frac{dz}{z}$ where C^- is the path defined in the proof of Proposition 2.2. The proof can be completed in the same way as in (2.2). □

For a primitive Dirichlet character χ mod f, the *k-th generalized Bernoulli number*, $B_{k,\chi}$ is defined as

$$F_\chi(t) = \sum_{a=1}^{f} \chi(a) \frac{t e^{at}}{e^{ft}-1} = \sum_{k=0}^{\infty} B_{k,\chi} \frac{t^k}{k!}.$$

Note that

$$\begin{aligned}\sum_{k=0}^{\infty} B_{k,\chi}\frac{t^k}{k!} &= \sum_{a=1}^{f}\chi(a)\frac{te^{at}}{e^{ft}-1} = \sum_{a=1}^{f}\chi(f-a)\frac{te^{(f-a)t}}{e^{ft}-1}\\ &= \chi(-1)\sum_{a=1}^{f}\chi(a)\frac{(-t)e^{-at}}{e^{-ft}-1} = \chi(-1)\sum_{k=0}^{\infty}(-1)^k B_{k,\chi}\frac{t^k}{k!}.\end{aligned}$$

Thus, $B_{k,\chi}=0$ for $k \not\equiv \delta \pmod 2$ if $\chi \neq 1$.

Proposition 5.3. *For a nontrivial primitive Dirichlet character* χ mod f*, the* k*-th generalized Bernoulli polynomial,* $B_{k,\chi}$*, satisfies the following equation:*

$$B_{k,\chi} = f^{k-1}\sum_{a=1}^{f}\chi(a)B_k\left(\frac{a}{f}\right),$$

where $B_k(x)$ *is the* k*-th Bernoulli polynomial defined in the previous section.*
In particular, $B_{1,\chi} = \dfrac{1}{f}\displaystyle\sum_{a=1}^{f}\chi(a)a.$

Proof. In the definition of $B_{k,\chi}$, substitute t/f for t, and we have

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{B_{k,\chi}}{f^k}\frac{t^k}{k!} &= \sum_{a=1}^{f}\chi(a)\frac{(t/f)e^{(a/f)t}}{e^t-1}\\ &= \frac{1}{f}\sum_{a=1}^{f}\chi(a)\sum_{k=0}^{\infty}B_k\left(\frac{a}{f}\right)\frac{t^k}{k!}.\end{aligned}$$

Comparing the coefficients of the t^k-th term, we obtain the desired equation. $B_{1,\chi}$ follows from the fact $\sum_{a=1}^{f}\chi(a)=0$ (see Remark). □

Some other properties of the generalized Bernoulli numbers are proved in the appendix.

Theorem 5.4. *For a Dirichlet character* χ *and any positive integer* k*,*

$$L(1-k,\chi) = -\frac{B_{k,\chi}}{k}.$$

This can be shown by using the proof of Theorem 5.2 and the functional equations of the gamma function, and by the same way as Theorem 2.5.

Theorem 5.5. *Let* χ *be a primitive Dirichlet character of conductor* f*. For a positive integer* k *with* $k \equiv \delta$ mod 2*,*

$$L(k,\chi) = (-1)^{1+(k-\delta)/2}\frac{\tau(\chi)}{2i^{\delta}}\left(\frac{2\pi}{f}\right)^k\frac{B_{k,\overline{\chi}}}{k!}.$$

Proof. The functional equations of the gamma function will be used without proofs, again. (See [1], [2], etc. for the proofs.)
The funtiona equation for $L(s,\chi)$ shows that

$$L(s,\chi) = \frac{\tau(\chi)}{i^{\delta}\sqrt{f}}\left(\frac{\pi}{f}\right)^{s-1/2}\Gamma\left(\frac{1-s+\delta}{2}\right)\Gamma\left(\frac{s+\delta}{2}\right)^{-1}L(1-s,\overline{\chi}). \quad \text{(See (5.2).)}$$

Under the hypothesis $k \equiv \delta \pmod 2$, $(k+\delta)/2$ is a positive integer, and it follows by a property of the gamma function that $\Gamma\left(\dfrac{k+\delta}{2}\right) = \left(\dfrac{k+\delta}{2} - 1\right)!$. Also, another property of the gamma function shows

$$\Gamma(s) = \frac{\Gamma(s+1)}{s} = \cdots = \frac{\Gamma(s+n)}{s(s+1)\cdots(s+n-1)},$$

which gives that

$$\Gamma\left(\frac{1-k+\delta}{2}\right) = \frac{\Gamma(1/2)}{2^{(-k+\delta)/2}(1-k+\delta)(3-k+\delta)\cdots(-1)} = \frac{(-1)^{(k-\delta)/2}2^{k-\delta-1}\sqrt{\pi}\left(\frac{k-\delta}{2}-1\right)!}{(k-\delta-1)!}.$$

Applying these and also Theorem 5.4, $L(k,\chi)$ may be written as

$$L(k,\chi) = (-1)^{\frac{k-\delta}{2}+1}\frac{\tau(\chi)}{i^{\delta}}\left(\frac{\pi}{f}\right)^{k}\frac{2^{k-\delta-1}\left(\frac{k-\delta}{2}-1\right)!}{(k-\delta-1)!\left(\frac{k+\delta}{2}-1\right)!k}B_{k,\overline{\chi}}.$$

But a fraction part on the right hand side can be reduced to $2^k/k!$ for any $k \equiv \delta \pmod 2$, and the desired result is obtained. □

6. p-adic Dirichlet L-functions

Finally, we would like to construct p-adic L-functions in this section. This section is based on Iwasawa [5, Chapter 3]. Throughout this section, let K be a finite extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}_p}$ and $K[[x]] := \{A(x) = \sum_{i=0}^{\infty} a_i x^i : a_i \in K\}$ the set of all power series. We say that $A(x)$ converges at s to mean that $|a_i s^i|_p \to 0$ as $i \to \infty$. We start this section from an important property of power series:

Lemma 6.1. *Let $A(x)$, $B(x) \in K[[x]]$ be convergent in a neighborhood of 0 in $\overline{\mathbb{Q}_p}$. Suppose there exists a sequence, $\{s_i\}$, in $\overline{\mathbb{Q}_p}$ such that $s_i \neq 0$, $s_i \to 0$ as $i \to 0$, and $A(s_i) = B(s_i)$ for all i. Then $A(x) = B(x)$.*

Proof. Suppose $A(x) \neq B(x)$ and write $A(x) - B(x) = \sum_{n=0}^{\infty} c_n x^n$. Then, there must exist n such that $c_n \neq 0$. Let n_0 be the minimum of such n. Then one has $0 = A(s_i) - B(s_i) = \sum_{n \geq n_0} c_n s_i^n$ for all i, and

$$|-c_{n_0}|_p = \left| s_i^{-n_0}\sum_{n\geq n_0} c_n s_i^n \right|_p = |s_i|_p \left| \sum_{n\geq n_0} c_n s_i^{n-n_0+1} \right|_p.$$

Since $\left|\sum_{n\geq n_0} c_n s_i^{n-n_0+1}\right|_p$ is bounded, if we take $i \to \infty$ then we get $c_{n_0} = 0$. This contradicts to our assumption. □

Now, let us define $\|\sum_{i=0}^{\infty} a_i x^i\| := \sup_i |a_i|_p$ and $P_K := \{A(x) \in K[[x]] : \|A\| < \infty\}$. We claim that $\| \ \ \|$ is a norm on P_K, and P_K is complete in $\| \ \ \|$. The proofs are shown below.

Proposition 6.2. *$\| \ \ \|$ is a norm on P_K.*

Proof. Let $A = \sum_{i=0}^{\infty} a_i x^i$ and $B = \sum_{i=0}^{\infty} b_i x^i$.

(i) Clearly $\|A\| \geq 0$. Also, $\|A\| = 0 \Leftrightarrow \sup_i |a_i|_p = 0 \Leftrightarrow |a_i|_p = 0$ for all $i \Leftrightarrow A = 0$.

(ii) $\|A+B\| = \sup_i |a_i + b_i|_p \leq \sup_i (\max\{|a_i|_p, |b_i|_p\}) = \max(\|A\|, \|B\|)$.

(iii) $\|AB\| = \sup_i(|a_i b_i|_p) \leq (\sup_i |a_i|_p)(\sup_i |b_i|_p) = \|A\| \cdot \|B\|$. □

Lemma 6.3. *P_K is complete in the norm, $\| \quad \|$.*

Proof. Let $\{A_k(x)\}_{k=1}^{\infty}$ be Cauchy in P_K and write $A_k(x) = \sum_{n=0}^{\infty} a_{k,n} x^n$, with $a_{kn} \in K$ for all k and n.
Let $\epsilon > 0$. Since A_k is Cauchy, there exists N such that $\|A_k - A_l\| < \epsilon$ for all $k,\, l \geq N$. Therefore, by the definition of $\| \quad \|$, $|a_{k,n} - a_{l,n}|_p < \epsilon$ for all $k,\, l \geq N$ and for all n. Hence $\{a_{k,n}\}_{k=1}^{\infty}$ is Cauchy in K for all n. But then, since K is complete (this is because Q_p is complete), there exists a_n in K such that $\{a_{k,n}\}$ converges to a_n as k approaches ∞. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$. To complete the proof, we need to show that $A_k(x)$ converges to $A(x)$ and $A(x)$ is in P_K.
Suppose that $A_k(x)$ does not converge to $A(x)$. Then, there exists $\epsilon > 0$ and a subsequence, $\{a_{k_j,n}\}$, of $\{a_{k,n}\}$ such that $|a_{k_j,n_j} - a_{n_j}|_p \geq 2\epsilon$ for some n_j. For this ϵ, choose N so that $|a_{k,n} - a_{l,n}|_p \leq \epsilon$ for all $k,\, l > N$ and for all n. If we let $l > N$ and fix $k_j > N$, we have $|a_{l,n_j} - a_{n_j}|_p \geq |a_{k_j,n_j} - a_{n_j}|_p - |a_{l,n_j} - a_{k_j,n_j}|_p > \epsilon$. Thus, $\{a_{l,n_j}\}$ does not converge to a_{n_j} as $l \to \infty$, and it contradicts that $\{a_{k,n}\}$ converges to a_n as $k \to \infty$ for all n.
Finally, since $\sup_n |a_{k,n} - a_n| < \epsilon$ for some k large enough and $|a_{k,n}|_p$ is bounded, $|a_n|_p$ must be also bounded for all n, and thus $A(x) \in P_K$. □

Now, define $\binom{x}{n}$, for any non-negative integer n, to be the polynomial of degree n given by

$$\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}.$$

This is continuous on Z_p.

Proposition 6.4. *Let n be a non-negative integer, and write it to the base p, i.e. $n = \alpha_0 + \alpha_1 p + \cdots + \alpha_t p^t$, with $0 \leq \alpha_i \leq p-1$ for all i, and let $S_n := \sum_{i=0}^{t} \alpha_i$. Then,*

$$\left\| \binom{x}{n} \right\| \leq p^{(n-S_n)/(p-1)}.$$

Proof. Clearly $\left\| \binom{x}{n} \right\| \leq \left| \frac{1}{n!} \right|_p$, and so we wish to compute $|1/n!|_p$. We claim that $\mathrm{ord}_p(n!) = (n-S_n)/(p-1)$.

$$\begin{aligned} \mathrm{ord}_p(n!) &= \sum_{i=1}^{n} \mathrm{ord}_p(i) \\ &= \sum_{\substack{1 \leq i \leq n \\ p \| i}} 1 + \sum_{\substack{1 \leq i \leq n \\ p^2 \| i}} 2 + \cdots + \sum_{\substack{1 \leq i \leq n \\ p^t \| i}} t. \end{aligned} \tag{6.5}$$

Note that the number of positive integers less than or equal to n and divisible by p is $[n/p] = \alpha_1 + \alpha_2 p + \cdots \alpha_t p^{t-1}$. Similarly, the number of positive integers that are less than or equal to n and divisible by p^j is $[n/p^j] = \alpha_j + \alpha_{j+1}p \cdots + \alpha_t p^{t-j}$ for all $j = 2, 3, \cdots, t$. Hence (6.5) is equal to

$$\sum_{j=1}^{t}\left[\frac{n}{p^j}\right] = \sum_{i=1}^{t}\alpha_i + p\sum_{i=2}^{t}\alpha_i + p^2\sum_{i=3}^{t}\alpha_i + \cdots + p^{t-1}\alpha_t.$$

On the other hand,

$$\begin{aligned}
\frac{n-S_n}{p-1} &= \frac{1}{p-1}\left(\alpha_t(p^t-1) + \alpha_{t-1}(p^{t-1}-1) + \cdots + \alpha_1(p-1)\right) \\
&= \alpha_t\left(1+p+\cdots+p^{t-1}\right) + \alpha_{t-1}\left(1+p+\cdots+p^{t-2}\right) + \cdots + \alpha_1 \\
&= \sum_{i=1}^{t}\alpha_i + p\sum_{i=2}^{t}\alpha_i + \cdots + p^{t-1}\alpha_t.
\end{aligned}$$

This proves the claim, and so $|1/n!|_p = p^{-\mathrm{ord}_p(1/n!)} = p^{\mathrm{ord}_p(n!)} = p^{(n-S_n)/(p-1)}$. $\square$

Theorem 6.6. *Let $0 < r < p^{-1/(p-1)}$ and $A(x) = \sum_{i=0}^{\infty} a_i\binom{x}{i}$ with $a_i \in K$ and $|a_i| \leq Mr^i$ for some M, for all i. Then $A(x) \in P_K$ and the radius of convergence of $A(x)$ is at least $\left(rp^{1/(p-1)}\right)^{-1}$.*

Proof. Let $A_k(x) = \sum_{n=0}^{k} a_n\binom{x}{n}$. Then, $A_k(x)$ is a polynomial of degree k, so it can be also written as $A_k(x) = \sum_{n=0}^{\infty} a_{k,n}x^n$ with $a_{k,n} = 0$ for all $n > k$. By Proposition 6.5, we have

$$\left\|a_n\binom{x}{n}\right\| \leq |a_n|_p p^{(n-S_n)/(p-1)} \leq Mr^n p^{(n-S_n)/(p-1)} < M\left(rp^{1/(p-1)}\right)^n,$$

and so $A_k(x)$ is in P_K. Also, $\{A_k(x)\}_{k=1}^{\infty}$ is Cauchy because, for $l > k$, we have

$$\|A_l - A_k\| \leq \max_{k<n\leq l}\left(\|a_n\binom{x}{n}\|\right) < M\left(rp^{1/(p-1)}\right)^{k+1},$$

and this converges to 0 as $k,\ l \to \infty$. By the choice of $A_k(x)$, it is clear that $\{A_k(x)\}$ converges to $A(x)$, and the limit is in P_K by the completeness of P_K.
Now, we wish to prove the last assertion. Write $A(x)$ as $A(x) := \sum_{n=0}^{\infty} a_{0,n}x^n$. Then, by Lemma 6.3, $a_{k,n} \to a_{0,n}$ as $k \to \infty$ for each n. Let $n < k$, then

$$|a_{k,n}|_p = |a_{k,n} - a_{n-1,n}|_p \leq \|A_k - A_{n-1}\| < M\left(rp^{1/(p-1)}\right)^n,$$

and by taking $k \to \infty$, we obtain $|a_{0,n}|_p < M\left(rp^{1/p-1}\right)^n$. Hence, if $|x|_p < \left(rp^{1/p-1}\right)^{-1}$ then $|a_{0,n}x^n|_p \to 0$ as $n \to \infty$. $\square$

Theorem 6.7. *Let $\{b_i\}_{i=1}^{\infty} \subset K$ and define $c_n := \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}b_i$. If there exists $M > 0$ such that $|c_n|_p < Mr^n$ for some r with $0 < r < p^{-1/(p-1)}$, then $A(x) := \sum_{n=0}^{\infty} c_n\binom{x}{n}$ is in P_K and $A(k) = b_k$ for all $k = 1, 2, 3, \cdots$.*

Proof. By the previous theorem, $A(x)$ is well-defined, and is in P_K. So we only need to show that $A(k) = b_k$ for all k. Let us write $A(x) = \sum_{n=0}^{\infty} a_n x^n$. First, we note that for any positive integer k, $|k|_p \leq 1$, and that the radius of convergence of A is at least $\left(rp^{1/(p-1)}\right)^{-1} > 1$ by

the theorem above. Thus, $A(k)$ is well-defined for all $k = 1, 2, 3, \cdots$.
Let $A_i(x) := \sum_{n=0}^{i} c_n \binom{x}{n}$. We claim $A_i(k) = b_k$ for $k < i$. This can be proven as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} c_n \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} b_i \right) \frac{t^n}{n!} \\ &= \left(\sum_{n=0}^{\infty} b_n \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} \right), \end{aligned}$$

so by multiplying both sides by e^t, we get

$$\sum_{n=0}^{\infty} b_n \frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} c_n \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \right) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \binom{n}{j} c_j \right) \frac{t^n}{n!}.$$

Thus, $b_n = \sum_{j=0}^{n} c_j \binom{n}{j}$, and it follows that, if $i > k$, then $A_i(k) = \sum_{n=0}^{i} c_n \binom{k}{n} = \sum_{n=0}^{k} c_n \binom{k}{n} = b_k$.
Let $\alpha \in \overline{\mathbb{Q}_p}$ with $|\alpha|_p < \left(rp^{1/(p-1)}\right)^{-1}$. It is now enough to show that $|A_i(\alpha) - A(\alpha)|_p \to 0$ as $i \to \infty$, or equivalently $|a_{i,n}\alpha^n - a_n\alpha^n|_p \to 0$ as $i \to \infty$ for all n.
If $i < n$, $|a_{i,n}\alpha^n - a_n\alpha^n|_p = |a_n\alpha^n|_p < M\left(rp^{1/(p-1)}\right)^n |\alpha|_p^n < M\left(rp^{1/(p-1)}\right)^i |\alpha|_p^i$. (See the proof of the previous theorem.)
If $i \geq n$, then

$$\begin{aligned} |a_{i,n}\alpha^n - a_n\alpha^n| &= |a_{i,n} - a_n|_p |\alpha|_p^n \leq \sup_n |a_{i,n} - a_n|_p \cdot |\alpha|_p^n \\ &= M\left(rp^{1/(p-1)}\right)^{i+1} |\alpha|_p^n \quad \text{by the proof of the previous theorem} \\ &= \begin{cases} M\left(rp^{1/(p-1)}\right)^i |\alpha|_p^i & \text{if } |\alpha|_p > 1 \\ M\left(rp^{1/(p-1)}\right)^i & \text{if } |\alpha|_p \leq 1 \end{cases}. \end{aligned}$$

Thus, $|a_{i,n} - a_n|_p$ is bounded for all n and converges to 0 as $i \to \infty$. Hence $A(k) = \lim_{i\to\infty} A_i(k) = b_k$. □

We note that Lemma 6.1 guarantees the uniqueness of such a power series $A(x)$.

Let q be an integer such that:

$$q := \begin{cases} p & \text{if } p \neq 2 \\ 4 & \text{if } p = 2. \end{cases}$$

We will fix this notation for the rest of this section. Now, let a be an element in $\mathrm{Z}_p^\times$ and write $a = \sum_{i=0}^{\infty} \alpha_i p^i$ with $0 \leq \alpha_i \leq p-1$ and $\alpha_0 \neq 0$. First suppose $p \neq 2$. Since α_0 is not divisible by p, we have $\alpha_0^{p-1} \equiv 1 \pmod{p}$. Therefore we can identify α_0 with a primitive $(p-1)$-th root of unity. Let us $\omega(a)$ be such a $(p-1)$-th root of unity. (This is the Teichmüller representative of α_0.) If $p = 2$, take an element such that $a = 1 + \alpha_1 2 + \sum_{i=2}^{\infty} \alpha_i 2^i$ with $\alpha_i \in \{0, 1\}$, and similarly identify $1 + \alpha_1 2$ with $\{\pm 1\}$. Also, let $\langle a \rangle := a/\omega(a)$. In this way, any p-adic unit a can be written as $\omega(a)\langle a \rangle$. We also note that $\langle a \rangle$ is an element in $1 + q\mathrm{Z}_p$. Let us extend ω to Z_p, by setting $\omega(a) = 0$ for all $a \in \mathrm{Z}_p/\mathrm{Z}_p^\times$. Then, clearly this is a Dirichlet character of conductor q.

For any Dirichlet character, χ, of conductor f, we define $\chi_n := \chi \cdot \omega^{-n}$ where ω is a Dirichlet character defined as above and n is any positive integer. Let us say f_n is the conductor of χ_n. Then, f_n must be a factor of fq. But f must be also a factor of $f_n q$ since $\chi = \chi_n \cdot \omega^n$, and so f_n differs from f by only a power of p. Therefore, if a is a rational integer such that $(a,p) = 1$ then $(a,f) = (a,f_n)$, and it follows that $\chi_n(a) = \chi(a)\omega(a)^{-n}$ for any such a.

Let $K = \mathbb{Q}_p(\chi)$ and define

$$b_k := \left(1 - \chi_k(p)p^{k-1}\right) B_{k,\chi_k} \quad \text{and} \quad c_k := \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} b_i.$$

Also let $A_\chi(x) := \sum_{n=0}^{\infty} c_n \binom{x}{n}$. We define *p-adic Dirichlet L-functions* as:

$$L_p(s,\chi) := \frac{1}{s-1} A_\chi(1-s).$$

We claim that this converges in $\left\{ s \in \overline{\mathbb{Q}_p} : |s-1|_p < \left(p^{1/(p-1)}\right)^{-1} |q|_p \right\}$. However, it requires a bit more work to prove that this function is even well-defined.

Proposition 6.8. *In* $\mathbb{Q}_p(\chi)$, $B_{k,\chi} = \lim_{n\to\infty} \frac{1}{p^n f} S_{k,\chi}(p^n f)$, *where* $S_{k,\chi}(n) = \sum_{a=1}^{n} \chi(a) a^k$.

In order to prove this proposition, we will use the *k-th generalized Bernoulli polynomial*, which is defined as follows:

$$F_\chi(x,t) = F_\chi(t) e^{xt} = \sum_{z=1}^{f} \chi(a) B_{k,\chi}(x) \frac{t^k}{k!}.$$

Similar to the Bernoulli polynomials, one can show that $B_{k,\chi}(x) = \sum_{i=0}^{k} \binom{k}{i} B_{i,\chi} x^{k-i}$. Some other properties of the generalized Bernoulli polynomials are listed in the Appendix.
Now we are ready to prove the above proposition.

Proof. of Proposition 6.8. We are going to use a property of the generalized Bernoulli polynomials, which says $B_{k,\chi}(x) - B_{k,\chi}(x-f) = k \sum_{a=1}^{f} \chi(a)(a+x-f)^{k-1}$ for all $k \geq 0$. For the proof of this property, see the proposition A.2.(3).
We claim that $S_{k,\chi}(nf) = \frac{1}{k+1}\left(B_{k+1,\chi}(nf) - B_{k+1,\chi}(0)\right)$. In order to prove this, take $x = nf, (n-1)f, \cdots, f$ in the equation above, and we get

$$\begin{aligned} B_{k+1,\chi}(nf) - B_{k+1,\chi}((n-1)f) &= (k+1) \sum_{a=1}^{f} \chi(a) \left(a + (n-1)f\right)^k \\ &\vdots \\ B_{k+1,\chi}(f) - B_{k+1,\chi}(0) &= (k+1) \sum_{a=1}^{f} \chi(a) a^k. \end{aligned}$$

Taking the sum of each side shows that:

$$\begin{aligned} B_{k+1,\chi}(nf) - B_{k+1,\chi}(0) &= (k+1)\sum_{a=1}^{f}\chi(a)\sum_{i=0}^{n-1}(a+if)^k \\ &= (k+1)\sum_{a=1}^{f}\sum_{i=0}^{n-1}\chi(a)(a+if)^k \\ &= (k+1)\sum_{a=1}^{f}\sum_{i=0}^{n-1}\chi(a+if)(a+if)^k \\ &= (k+1)\sum_{a=1}^{nf}\chi(a)a^k = (k+1)S_{k,\chi}(nf), \end{aligned}$$

and this completes the proof of our claim.

Using our claim and the property, $B_{k,\chi}(x) = \sum_{i=0}^{k}\binom{k}{i}B_{i,\chi}x^{k-i}$,

$$S_{k,\chi}(p^n f) = \frac{1}{k+1}\sum_{i=1}^{k+1}\binom{k+1}{i}(p^n f)^{k+1-i}B_{i,\chi} = p^n f B_{k,\chi} + (p^n f)^2 \cdot (\text{other terms}),$$

or $S_{k,\chi}(p^n f)/(p^n f) = B_{k,\chi} + p^n f \cdot (\text{other terms})$. So take $n \to \infty$ and we get the desired result. □

Proposition 6.9. $c_k \equiv 0 \pmod{q^{k-2}f^{-1}}$ *for all* $k = 1, 2, 3, \cdots$

Proof. By the definition of b_k and Proposition 6.8, one can see that

$$\begin{aligned} b_k &= \lim_{n\to\infty}\frac{1}{p^n f}\sum_{a=1}^{p^n f}\chi_k(a)a^k - \lim_{n\to\infty}\frac{\chi_k(p)p^{k-1}}{p^n f}\sum_{a=1}^{p^n f}\chi_k(a)a^k \\ &= \lim_{n\to\infty}\frac{1}{p^n f}\sum_{a=1}^{p^n f}\chi_k(a)a^k - \lim_{n\to\infty}\frac{\chi_k(p)p^{k-1}}{p^{n-1} f}\sum_{a=1}^{p^{n-1} f}\chi_k(a)a^k \\ &= \lim_{n\to\infty}\frac{1}{p^n f}\left(\sum_{a=1}^{p^n f}\chi_k(a)a^k - \sum_{a=1}^{p^{n-1} f}\chi_k(pa)(pa)^k\right) \\ &= \lim_{n\to\infty}\frac{1}{p^n f}\sum_{\substack{a=1\\ a\in\mathbb{Z}_p^\times}}^{p^n f}\chi_k(a)a^k = \lim_{n\to\infty}\frac{1}{p^n f}\sum_{\substack{a=1\\ a\in\mathbb{Z}_p^\times}}^{p^n f}\chi(a)\omega(a)^{-k}\omega(a)^k\langle a\rangle^k \\ &= \lim_{n\to\infty}\frac{1}{p^n f}\sum_{\substack{a=1\\ a\in\mathbb{Z}_p^\times}}^{p^n f}\chi(a)\langle a\rangle^k. \end{aligned}$$

Hence

$$c_k = \lim_{n\to\infty}\frac{1}{p^n f}\sum_{i=0}^{k}\sum_{\substack{a=1\\ a\in\mathbb{Z}_p^\times}}^{p^n f}\binom{k}{i}(-1)^{k-i}\chi(a)\langle a\rangle^i = \lim_{n\to\infty}\frac{1}{p^n f}\sum_{\substack{a=1\\ a\in\mathbb{Z}_p^\times}}^{p^n f}\chi(a)\left(\langle a\rangle - 1\right)^k.$$

So we need to show that $\frac{1}{q^n f}\sum_{\substack{a=1\\ a\in \mathbb{Z}_p^\times}}^{q^n f}\chi(a)\left(\langle a\rangle-1\right)^k\equiv 0 \pmod{q^{k-2}f^{-1}}$ for all $n\geq 1$. We wish to show it by induction.

First let $n=1$. Then the left hand side of the congruence is congruent to 0 modulo $(q^{k-1}f^{-1})$ since $(\langle a\rangle-1)^k\equiv 0 \pmod{q^k}$, and so is congruent to 0 modulo $(q^{k-2}f^{-1})$.

Now let $n>1$, and a a positive integer less than $q^{n+1}f$ with $(a,p)=1$. Write $a=\alpha_0+\alpha_1 q^n f$ with $0\leq \alpha_0\leq q^n f-1$ and $0\leq \alpha_1\leq q-1$. This α_0 satifies $\omega(a)=\omega(\alpha_0)$ and $\chi(a)=\chi(\alpha_0)$ because $a\equiv\alpha_0 \pmod{q}$, and

$$\begin{aligned} a=\omega(\alpha)\langle a\rangle &= \omega(\alpha_0)\langle\alpha_0\rangle+\omega(\alpha_1 q^n f)\langle\alpha_1 q^n f\rangle\\ &= \omega(a)\langle\alpha_0\rangle+1\cdot\alpha_1 q^n f\end{aligned}$$

This shows that $\langle a\rangle=\langle\alpha_0\rangle+\omega(a)^{-1}\alpha_1 q^n f$, and so we obtain

$$\begin{aligned} (\langle a\rangle-1)^k &= \left(\langle\alpha_0\rangle-1+\omega(a)^{-1}\alpha_1 q^n f\right)^k\\ &= \sum_{i=0}^{k}\binom{k}{i}\left(\langle\alpha_0-1\right)^i\left(\omega(a)^{-1}\alpha_1 q^n f\right)^{k-i}.\end{aligned}$$

But then since $\left(\langle\alpha_0\rangle-1\right)^i\equiv 0 \pmod{q^i}$ and $\left(\omega(a)^{-1}\alpha_1 q^n f\right)^{k-i}\equiv 0 \pmod{q^{n(k-i)}}$, the i-th term of the sum is zero mod $(q^{i+n(k-i)})$. It follows that $\left(\langle a\rangle-1\right)^k\equiv\left(\langle\alpha_0-1\right)^k \pmod{q^{n+k-1}}$ since, for any $i<k$, $\;i+n(k-i)=(n-1)(k-i)+k\geq n+k-1$. Multiplying by χ and taking the sum give

$$\sum_{\substack{a=1\\ p\nmid a}}^{q^{n+1}f}\chi(a)\left(\langle a\rangle-1\right)^k\equiv q\sum_{\substack{\alpha_0=1\\ p\nmid\alpha_0}}^{q^n f}\chi(\alpha_0)\left(\langle\alpha_0-1\rangle-1\right)^k \pmod{q^{n+k-1}},$$

or equivalently

$$\frac{1}{q^{n+1}f}\sum_{\substack{a=1\\ p\nmid a}}^{q^{n+1}f}\chi(a)\left(\langle a\rangle-1\right)^k\equiv\frac{1}{q^n f}\sum_{\substack{a=1\\ p\nmid a}}^{q^n f}\chi(a)\left(\langle a\rangle-1\right)^k \pmod{q^{k-2}f^{-1}}.$$

This is zero mod $(q^{k-2}f^{-1})$ by the induction hypothesis. Take $n\to\infty$, and we get the desired result. $\square$

This proposition says $|c_k|_p\leq\left|q^{-2}f^{-1}\right|_p|q|_p^k$. Thus, by taking $r=|q|_p$ $(<p^{-1/(p-1)})$ and $M=|q^{-2}f^{-1}|_p$ in Theorem 6.7, one can show that A_χ is well-defined in $P_{\mathbb{Q}_p(\chi)}$ and that it converges at s for $\left\{s\in\overline{\mathbb{Q}_p}:|s-1|_p<\left(p^{1/(p-1)}|q|_p\right)^{-1}\right\}$.

Proposition 6.10. *For a Dirichlet character χ and any positive integer k,*

$$L_p(1-k,\chi)=\left(1-\chi_k(p)p^{k-1}\right)\left(-\frac{B_{k,\chi_k}}{k}\right).$$

Proof. This follows directly from the definition and Proposition 6.7, because

$$L_p(1-k,\chi)=-\frac{1}{k}A_\chi(k)=-\frac{1}{k}\left(1-\chi_k(p)p^{k-1}\right)B_{k,\chi_k}.$$

□

We also note that this proposition and Lemma 6.1 prove the uniquness of such a meromorphic function. Moreover, since $A_\chi(x) \in P_{\mathbb{Q}_p(\chi)}$, one can write A_χ as a power series, say $A_\chi(x) = \sum_{n=0}^{\infty} a_{n-1}x^n$. Then,

$$\begin{aligned} L_p(s,\chi) &= \frac{1}{s-1}\sum_{n=0}^{\infty} a_{n-1}(1-s)^n = \frac{1}{s-1}\sum_{n=0}^{\infty}(-1)^n a_{n-1}(s-1)^n \\ &= \frac{a_{-1}}{s-1} + \sum_{n=0}^{\infty}(-1)^{n+1}a_n(s-1)^n. \end{aligned}$$

If $s = 1$, then we have $A_\chi(0) = (1-\chi(p)p^{-1})B_{0,\chi}$ since $\chi_0 = \chi$. If $\chi \neq 1$, then $B_{0,\chi} = 0$ by Proposition 4.4 and Remark 1, and so $A_\chi(0) = 0$. Therefore, $a_{-1} = 0$ and $L_p(s,\chi)$ is holomorphic at $s = 1$. If $\chi = 1$, then $B_{0,\chi} = 1$ and so $A_\chi(0) = 1 - 1/p$. Hence $L_p(s, 1)$ has a pole at $s = 1$ and its residue is $1 - 1/p$.

Appendix A. Bernoulli Numbers

In this appendix, we will prove some properties of Beroulli polynomials and generalized Bernoulli polynomials. First recall each definition.

Definition A.1.

$$\text{Bernoulli numbers:} \quad \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \frac{t}{e^t - 1}$$

$$\text{Bernoulli polynomials:} \quad \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!} = \frac{te^{xt}}{e^t - 1}$$

$$\text{Generalized Bernoulli numbers:} \quad \sum_{k=0}^{\infty} B_{k,\chi} \frac{t^k}{k!} = \sum_{a=1}^{f} \chi(a) \frac{te^{at}}{e^{ft} - 1}$$

$$\text{Generalized Bernoulli polynomials:} \quad \sum_{k=0}^{\infty} B_{k,\chi}(x) \frac{t^k}{k!} = \sum_{a=1}^{f} \chi(a) \frac{te^{(a+x)t}}{e^{ft} - 1}$$

Proposition A.2. *For the Bernoulli polynomials, the followings are true.*

(1) $$B_k(x) = \sum_{j=0}^{k} \binom{k}{j} B_j x^{k-j}$$

Proof. Consider the following.

$$\begin{aligned} \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!} &= \frac{te^{xt}}{e^t - 1} \cdot e^{xt} = \left(\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} \right) \left(\sum_{k=0}^{\infty} \frac{x^k t^k}{k!} \right) \\ &= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j} B_j x^{k-j} \right) \frac{t^k}{k!}. \end{aligned}$$

□

(2) $$B_k(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} B_j x^{k-j}$$

Proof. Note that

$$\frac{t}{e^t - 1} = \frac{te^{-1}}{1 - e^{-t}} = \sum_{k=0}^{\infty} (-1)^k B_k \frac{t^k}{k!}.$$

Therefore,

$$\begin{aligned} \frac{t}{e^t - 1} \cdot e^{xt} &= \left(\sum_{k=0}^{\infty} (-1)^k B_k \frac{t^k}{k!} \right) \left(\sum_{k=0}^{\infty} \frac{x^k t^k}{k!} \right) \\ &= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} (-1)^j \frac{B_j}{j!} \frac{x^{k-j}}{(k-j)!} \right) t^k \\ &= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} (-1)^j \binom{k}{j} B_j x^{k-j} \right) \frac{t^k}{k!}. \end{aligned}$$

□

(3) $$B_k(x+1) - B_k(x) = kx^{k-1} \qquad (k \geq 0)$$

Proof. Consider the following,

$$\sum_{k=0}^{\infty} (B_k(x+1) - B_k(x)) \frac{t^k}{k!} = \frac{te^{xt}}{e^t - 1}(e^t - 1) = \sum_{k=1}^{\infty} \frac{x^k t^{k+1}}{k!},$$

and compare the coefficients. □

(4) $$B_k(1-x) = (-1)^k B_k(x) \qquad (k \geq 0)$$

Proof.

$$\begin{aligned}
\sum_{k=0}^{\infty} B_k(1-x)\frac{t^k}{k!} &= \frac{te^t}{e^t-1}e^{-xt} = \frac{-t}{e^{-t}-1}e^{-xt} \\
&= \left(\sum_{k=0}^{\infty}(-1)^k B_k \frac{t^k}{k!}\right)\left(\sum_{k=0}^{\infty}\frac{(-x)^k t^k}{k!}\right) \\
&= \sum_{k=0}^{\infty}(-1)^k \left(\sum_{j=0}^{k}\binom{k}{j} B_j x^{k-j}\right)\frac{t^k}{k!}
\end{aligned}$$

□

(5) $$B'_k(x) = kB_{k-1}(x)$$

Proof. Use the second property to get

$$B'_k(x) = \sum_{j=0}^{k-1}(-1)^j \binom{k}{j} B_j x^{k-j-1}(k-j),$$

and apply $(k-j)\binom{k}{j} = k\binom{k-1}{j}$. Then

$$B'_k(x) = k\sum_{j=0}^{k-1}(-1)^j \binom{k-1}{j} B_j x^{k-j-1}.$$

□

(6) $$B_k(0) = B_k, \quad \textit{and} \quad B_k(1) = (-1)^k B_k \qquad (k \geq 0)$$

Proof. For the first part, take $x = 0$ in (1), and for the second part, take $x = 0$ in (4). □

(7) $$\int_0^1 B_k(x)dx = 0$$

Proof.

$$\begin{aligned}
\int_0^1 B_k(x)dx &= \frac{1}{k+1}\int_0^1 B'_{k+1}(x)dx \qquad \text{by (5)} \\
&= \frac{1}{k+1}\left(B_{k+1}(1) - B_{k+1}(0)\right) = 0 \qquad \text{by (6).}
\end{aligned}$$

□

Proposition A.3. *For the generalized Bernoulli polynomials, the followings are true.*

$$B_{k,\chi}(x) = \sum_{j=0}^{k} \binom{k}{j} B_{j,\chi} x^{k-j} \tag{1}$$

Proof. Consider the following.

$$\begin{aligned}
\sum_{k=0}^{\infty} B_{k,\chi}(x)\frac{t^k}{k!} &= \sum_{a=1}^{f} \chi(a)\frac{te^{at}}{e^{ft}-1} \cdot e^{xt} = \left(\sum_{k=0}^{\infty} B_{k,\chi}\frac{t^k}{k!}\right)\left(\sum_{k=0}^{\infty} \frac{x^k t^k}{k!}\right) \\
&= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j} B_{k,\chi} x^{k-j}\right) \frac{t^k}{k!}
\end{aligned}$$

□

$$B_{k,\chi}(x) = \chi(-1)\sum_{j=0}^{k}(-1)^j \binom{k}{j} B_{j,\chi} x^{k-j} \tag{2}$$

Proof. First, note that

$$\begin{aligned}
\sum_{a=1}^{f} \chi(a)\frac{te^{at}}{e^{ft}-1} &= \sum_{a=1}^{f} \chi(f-a)\frac{te^{-at}}{1-e^{-ft}} \\
&= \chi(-1)\sum_{a=1}^{f} \chi(a)\frac{(-t)e^{-at}}{e^{-ft}-1} \\
&= \chi(-1)\sum_{k=0}^{\infty}(-1)^k B_{k,\chi}\frac{t^k}{k!}.
\end{aligned}$$

Since $\sum_{k=0}^{\infty} B_{k,\chi}(x)\frac{t^k}{k!} = \sum_{a=1}^{f} \chi(a)\frac{te^{at}}{e^{ft}-1}e^{xt}$, now apply the previous result for the right hand side, and follow the same argument as of (1) to get the desired result. □

$$B_{k,\chi}(x) - B_{k,\chi}(x-f) = k\sum_{a=1}^{f} \chi(a)(a+x-f)^{k-1} \tag{3}$$

Proof. Consider the following.

$$\begin{aligned}
\sum_{k=0}^{\infty} \left(B_{k,\chi}(x) - B_{k,\chi}(x-f)\right)\frac{t^k}{k!} &= \sum_{a=1}^{f} \chi(a)\frac{te^{(a+x)t}}{e^{ft}-1}(1-e^{-ft}) \\
&= \sum_{a=1}^{f} \chi(a)\frac{te^{(a+x)t}e^{-ft}}{e^{ft}-1}(e^{ft}-1) \\
&= \sum_{a=1}^{f} \chi(a)te^{(a+x-f)t} \\
&= \sum_{a=1}^{f} \chi(a)t\sum_{k=0}^{\infty} \frac{(a+x-f)^k t^k}{k!} \\
&= \sum_{k=0}^{\infty}\sum_{a=1}^{f} \chi(a)(a+x-f)^k \frac{t^{k+1}}{k!}.
\end{aligned}$$

This shows that

$$\begin{aligned} B_{k,\chi}(x) - B_{k,\chi}(x-f) &= \sum_{a=1}^{f} \chi(a)(a+x-f)^{k-1} \frac{k!}{(k-1)!} \\ &= k \sum_{a=1}^{f} \chi(a)(a+x-f)^{k-1} \quad \text{for} \quad k \geq 0. \end{aligned}$$

□

(4) $$B_{k,\chi}(f-x) = \chi(-1) \sum_{j=0}^{k} (-1)^j f^{k-j} \binom{k}{j} B_{j,\chi}(x)$$

Proof.

$$\begin{aligned} \sum_{k=0}^{\infty} B_{k,\chi}(f-x) \frac{t^k}{k!} &= \chi(-1) \left(\sum_{k=0}^{\infty} (-1)^k B_{k,\chi} \frac{t^k}{k!} \right) e^{-xt} e^{ft} \\ &= \chi(-1) \left(\sum_{k=0}^{\infty} (-1)^k B_{k,\chi} \frac{t^k}{k!} \right) \left(\sum_{k} \frac{(-1)^k x^k t^k}{k!} \right) \left(\sum_{k} \frac{f^k t^k}{k!} \right) \\ &= \chi(-1) \left(\sum_{k=0}^{\infty} (-1)^k \left(\sum_{i=0}^{k} \binom{k}{i} B_{i,\chi} x^{k-i} \right) \frac{t^k}{k!} \right) \left(\sum_{k=0}^{\infty} \frac{f^k t^k}{k!} \right) \\ &= \chi(-1) \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} (-1)^j B_{j,\chi}(x) \binom{k}{j} f^{k-j} \right) \frac{t^k}{k!}. \end{aligned}$$

□

(5) $$B'_{k,\chi}(x) = kB_{k-1,\chi}(x)$$

Proof. Using the second property, we get

$$B'_{k,\chi}(x) = \chi(-1) \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} B_{j,\chi} x^{k-j-1} (k-j)$$

Apply the property $(k-j)\binom{k}{j} = k\binom{k-1}{j}$ to get

$$B'_{k,\chi}(x) = \chi(-1) \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} B_{j,\chi} x^{k-j-1} \cdot k = kB_{k-1,\chi}(x).$$

□

(6) $$B_{k,\chi}(0) = B_{k,\chi}, \qquad (k \geq 0), \quad \textit{and} \quad B_{k,\chi}(f) = k \sum_{a=1}^{f} \chi(a) a^{k-1} + B_{k,\chi}, \qquad (k \geq 1)$$

Proof. For the first part, take $x = 0$ in (1), and for the second part, take $x = f$ in (3). □

(7) $$\int_0^f B_{k,\chi}(x) dx = \sum_{a=1}^{f} \chi(a) a^k$$

Proof.

$$\begin{aligned}\int_0^f B_{k,\chi}(x)dx &= \frac{1}{k+1}\int_0^f B'_{k+1,\chi}(x)dx \qquad \text{by (5)}\\ &= \frac{1}{k+1}(B_{k+1,\chi}(f)-B_{k+1,\chi}(0))\\ &= \frac{1}{k+1}\left((k+1)\sum_{a=1}^{f}\chi(a)a^k + B_{k+1,\chi}-B_{k+1,\chi}\right) \qquad \text{by (6)}\\ &= \sum_{a=1}^{f}\chi(a)a^k.\end{aligned}$$

□

Theorem A.4. *(Clausen and von Staudt)*

(1) *If* $p>2$ *and* $(p-1)|k$, *then* $pB_k \equiv -1 \pmod p$.

(2) *If* $p=2$ *and* k *is even or* 1, *then* $2B_k \equiv 1 \pmod 2$.

Proof. To prove the first statement, let $\alpha = 1+p$ and consider

$$pB_k = -kp\left(-\frac{B_k}{k}\right) = -kp(1-p^{k-1})^{-1}(\alpha^{-k}-1)^{-1}\int_{\mathbb{Z}_p^\times} x^{k-1}\mu_{1,\alpha}. \tag{A.5}$$

Since $\alpha^{-k}-1 = (1+p)^{-k}-1 \equiv -kp \pmod{p^{\mathrm{ord}_p k+2}}$, $-kp/(\alpha^{-k}-1) \equiv 1 \pmod{p^{\mathrm{ord}_p k+2}}$, and so this is congruent to 1 mod$p$. Therefore (A.4) is congruent to $\int_{\mathbb{Z}_p^\times} x^{k-1}\mu_{1,\alpha}$ mod p. Now we claim that $\int_{\mathbb{Z}_p^\times} x^{k-1}\mu_{1,\alpha} \equiv \int_{\mathbb{Z}_p^\times} x^{-1}\mu_{1,\alpha} \pmod p$ for any p-adic unit, x. This can be proven by mimicking the first paragraph of Section 4. To complete the proof, we claim that $\int_{\mathbb{Z}_p^\times} x^{-1}\mu_{1,\alpha} \equiv -1 \pmod p$. For any $x \in \mathbb{Z}_p^\times$, write $x = \sum_{i=0}^\infty x_i p^i$ with $x_0 \neq 0$, and define $g(x) = 1/x_0$. Note that this is a locally constant function, for $g(x_0 + p\mathbb{Z}_p) = 1/x_0$. Repeating the same argument as the proof of the previous claim, we get $\int_{\mathbb{Z}_p^\times} x^{-1}\mu_{1,\alpha} \equiv \int_{\mathbb{Z}_p^\times} g\mu_{1,\alpha} \pmod p$, and it is now enough to show the right hand side of the last equality is congruent to -1 mod p.
Write g as $g = \sum_{a=1}^{p-1}\chi_{a+(p)}/a$ where $\chi_{a+(p)}(x) = 1$ if $x \in a+(p)$ and 0 otherwise, and we obtain

$$\int_{\mathbb{Z}_p^\times} g\mu_{1,\alpha} = \sum_{a=1}^{p-1}\frac{1}{a}\int_{a+(p)} 1\mu_{1,\alpha} = \sum_{a=1}^{p-1}\frac{1}{a}\mu_{1,\alpha}(a+(p)).$$

Simplify the right hand side by using the definition of $\mu_{1,\alpha}$ and substituting $\alpha = a+p$, the desired result follows. This also completes the proof of the theorem.

For the second part, if $k=1$ or 2 then clearly $2B_k \equiv 1 \pmod 2$. So now suppose $k \geq 4$ and k is even, and let $\alpha = 5$. The same proof as above shows that $2B_k \equiv 1/2\int_{\mathbb{Z}_2^\times} x^{k-1}\mu_{1,5} \pmod{2^2} \equiv 1/2\int_{z_2^\times} x^{-1}\mu_{1,5} \pmod{2^2}$. Now, we claim that $\int_{\mathbb{Z}_2^\times} x^{-1}\mu_{1,5} \equiv 2 \pmod{2^2}$. This can be shown in the same manner as the first part, with a locally constant function $g(x) = 1/(1+x_1\cdot 2)$ for any 2-adic unit $x = 1+\sum_{i=1}^\infty x_i\cdot 2^i$. This gives that $2B_k \equiv 1 \pmod 4$ and so it is congruent to 1 mod2.

□

References

[1] L.V. Ahlfors, *Complex Analysis.* An introduction to the theory of analytic functions of one complex variable. Third edition. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York, 1978.

[2] T. Arakawa, T. Ibukiyama, and M. Kaneko, *Bernoulli numbers and Zeta-Functions.* Makino-Shoten, Tokyo, 2001. (Japanese)

[3] D. Bump, *Automorphic Forms and Representations.* Cambridge Studies in Advanced Mathematics, 55. Cambridge University Press, Cambridge, 1997.

[4] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory.* Revised edition of Elements of number theory. Graduate Texts in Mathematics, 84. Springer-Verlag, New York-Berlin, 1982.

[5] K. Iwasawa, *Lectures on p-adic L-Functions.* Annals of Mathematics Studies, No. 74. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1972.

[6] N. Koblitz, *p-adic Numbers, p-adic Analysis, and Zeta-Functions.* Second edition. Graduate Texts in Mathematics, 58. Springer-Verlag, New York, 1984.

[7] J. Neukirch, *Algebraic number theory.* Translated from the 1992 German original and with a note by Norbert Schappacher. With a foreword by G. Harder. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 322. Springer-Verlag, Berlin, 1999.

[8] L.C. Washington, *Introduction to Cyclotomic Fields.* Graduate Texts in Mathematics, 83. Springer-Verlag, New York, 1982.