Computing β-Drawings of 2-Outerplane Graphs
(Extended Abstract)

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Abstract. A proximity drawing of a plane graph G is a straight-line drawing of G with the additional geometric constraint that two vertices of G are adjacent if and only if the well-defined "proximity region" of these two vertices does not contain any other vertex. In one class of proximity drawings, known as β-drawings, the proximity region is parameterized by β, where $\beta \in [0, \infty]$. Given a plane graph G, the "β-drawability problem" asks whether G has a β-drawing Γ and in that case G is called "β-drawable." The problem of whether a class of graphs is β-drawable, for some value of β, has been studied for two classes of graphs– trees and outerplanar graphs. However, for larger classes of graphs the β-drawability problem is still an open problem. In this paper we focus on the problem of β-drawability of 2-outerplane graphs for $1 < \beta < 2$. We provide a set of sufficient conditions for a biconnected 2-outerplane graph to have a β-drawing, for $1 < \beta < 2$. We provide a drawing algorithm as well. We also identify a subclass of biconnected 2-outerplane graphs that are not β-drawable for $1 < \beta < 2$.

Keywords: Proximity graphs, Graph drawing, Proximity drawings of graphs, β-drawings.

1 Introduction

Geometric graphs such as Voronoi diagrams, Delaunay triangulations, convex hulls, visibility graphs, etc. have evolved over time for modeling and solving various practical problems. Proximity graphs are another class of geometric graphs that has received a great deal of interest from the computational geometry community. Proximity graphs have been used in a wide range of application areas. In this section, first we introduce proximity graphs. Then we specify several application areas where proximity graphs are being used. We introduce a parameterized family of proximity graphs, known as the β-proximity graphs, for $0 \leq \beta \leq \infty$. Then we review the literature and finally, state the results of the paper.

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1.1 Proximity Graphs

Even though the notion of "proximity graphs" was coined in much later, essentially the foundation of this has been the *Gabriel graph*, introduced by Gabriel and Sokal in the context of geographic variation analysis [GS69]. A Gabriel graph is a plane graph in which two vertices are adjacent if and only if the closed circle having these two vertices as its two antipodal points contains no other vertex of the graph. Here the closed circle just mentioned is also known as the *proximity region*, specifically the *Gabriel region*, of the two vertices. Like Gabriel graphs other types of proximity graphs also have a well-defined proximity region. Proximity regions are also termed as *regions of influence* by some authors. A definition of the proximity region is at the heart of a proximity graph. All the geometric properties of any proximity graph are just results of the definition of its proximity region. For different proximity regions we get different proximity graphs although they might have the same set of points in the plane, each point being represented by a vertex of the corresponding graph. We clarify this idea by an example of constructing a proximity graph from a point set.

Let us consider a set of points in 2-dimensional space, as illustrated in Fig. 1(i). Taking these points as the vertices, we can compute the corresponding Gabriel graph by adding an edge between any two distinct vertices if and only if the proximity region, i.e. the closed circle having the points corresponding to these two vertices as its antipodal points, does not contain any other point. The corresponding Gabriel graph is shown in Fig. 1(ii).

Another type of proximity graph is the relative neighborhood proximity graph. In this type of proximity graph the proximity region of two points x and y is the intersection of two open disks of radius $d(x, y)$ centered at x and y. In a relative neighborhood graph there is an edge between x and y if and only if the relative neighborhood proximity region of x and y is empty. Fig. 1(iii) shows the relative neighborhood graph resulting from the point set shown in Fig. 1(i).

As shown above, given a set of points and a definition of the proximity region we can construct a proximity graph. An important property of this graph is that it provides us a description of the internal structure of the set of points. For some other definition of the proximity region, namely the γ-proximity region, the resulting proximity graph can describe the external shape of the point set [Velt92]. The ability of proximity graphs to describe the internal or external structure of a set of points has found its application in *computational morphology*, which is concerned with the analysis of the shape of a set of points. Research findings in computational morphology have industrial applications in *computer vision*. Proximity graphs have also been used in graph-based methods of clustering and manifold learning [CZ05]. Besides that, proximity graph-based methods have been applied in data mining [Tous05], topology control in wireless sensor networks [Li03] and in many other diverse fields.

We have just seen two types of proximity graphs: the Gabriel graph and the relative neighborhood graph. One might wonder how many other types of proximity graphs are there. In fact, there is an infinite number of different types of proximity graphs. For example, an infinite family of parameterized proximity

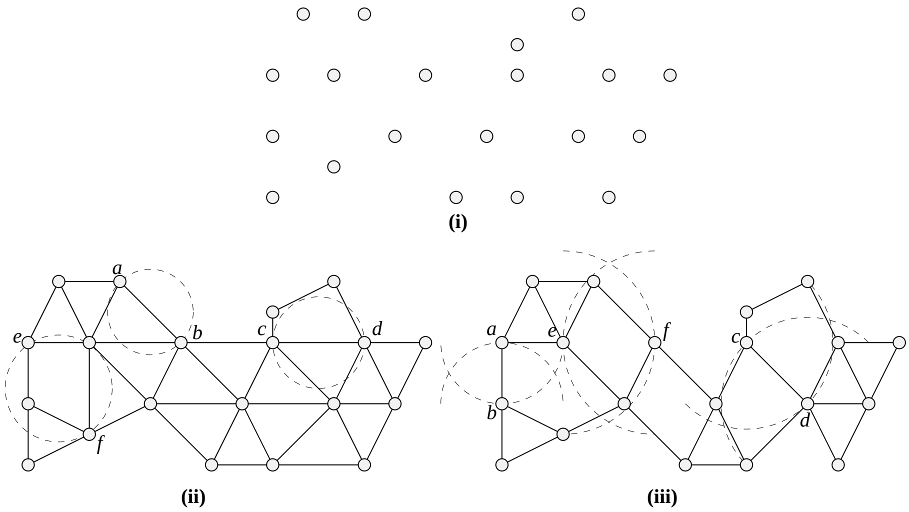

Fig. 1. (i) A set of points (ii)Gabriel graph and (iii) relative neighborhood graph corresponding to the set of points shown in (i). The dotted regions explain why the edges (a,b) and (c,d) have been added and why the edge (e,f) has not been added.

graphs have been introduced by Kirkpatrick and Radke [KR85]. This family of proximity graphs is called β-skeletons, where β stands for the parameter that can take any real number value in $[0, \infty]$. Interestingly, Gabriel graph and relative neighborhood graph both belong to this family of proximity graphs. Gabriel graph is the closed proximity graph for the value of $\beta = 1$ and relative neighborhood graph is the open proximity graph for $\beta = 2$.

1.2 Literature Review

Being brought to light in the year of 1969, proximity graphs might seem to be old geometric graphs. But the most interesting thing is that it has been providing new research trends quite regularly. The oldest research direction concerning proximity graphs is: given a set of points and a definition of the proximity region, how can we compute the proximity graph efficiently and what are the properties of this graph? This research area has been explored and reviewed very nicely in a paper by Jaromczyk and Toussaint [JT92].

Apart from the problem of computing a proximity graph and analyzing its underlying properties, another problem that has been receiving a lot of attention from the graph drawing community of late is: given a planar embedding of a graph and a definition of the proximity region, is it possible to achieve a *straight-line drawing* of the graph maintaining the proximity constraints? If yes then how can it be drawn? This has emerged as one of the relatively new graph drawing challenges. Today it is widely known as the *proximity drawability problem*. Intuitively, the initial research effort in this direction has focused on proximity drawability of trees. The problem of proximity drawability of trees can be framed in this way: given a class of trees and a value of β, does this class of trees admit a β-drawing? Here we can classify the trees according to the maximum vertex degree. This problem has been studied in [BDLL95,BLL96] and the classes of trees that admit open β-drawings for $0 \leq \beta \leq \frac{1}{1-cos(\frac{2\pi}{5})}$ and $\frac{1}{cos(\frac{2\pi}{5})} < \beta < \infty$ have been found. In the same paper the classes of trees that admit closed β-drawings for $0 \leq \beta < \frac{1}{1-cos(\frac{2\pi}{5})}$ and $\frac{1}{cos(\frac{2\pi}{5})} \leq \beta \leq \infty$ have also been been found. Outside these ranges of β values, for $\beta = 2$ it has been found that the class of finite trees with maximum vertex degree at most 5 are β-drawable [BLL96]. However, there is some gray area for $\frac{1}{1-cos(\frac{2\pi}{5})} \leq \beta \leq \frac{1}{cos(\frac{2\pi}{5})}$ and $\beta \neq 2$ in the sense that determining which classes of trees are β-drawable for this range of β values is still an open problem.

Another class of graphs that has been studied from the graph drawing perspective is the class of outerplanar graphs. It has been shown that all biconnected outerplanar graphs can be β-drawn for $1 \leq \beta \leq 2$ [LL96]. In the same paper it has been shown that β-drawability of connected outerplanar graphs for some value of $\beta \in [0, \infty]$ depends on the maximum vertex degree in the block-cut-vertex tree of that graph. One of the open problems left in [LL96] is to extend the problem of β-drawability of graphs to other nontrivial classes of graphs apart from trees and outerplanar graphs. In this paper we characterize a subclass of biconnected 2-outerplane graphs that can be β-drawn for $\beta \in (1, 2)$. We also show that all biconnected 2-outerplane graphs are not β-drawable for this range of β values.

1.3 Results

In this paper we are concerned with the problem of β-drawability of biconnected 2-outerplane graphs. The results of the paper are summarized as follows:

- We have specified a necessary condition for β-drawability of biconnected 2-outerplane graphs, for $1 < \beta < 2$.
- We have specified sufficient conditions for β-drawability of biconnected 2-outerplane graphs, for $1 < \beta < 2$. The sufficient conditions induce a large and nontrivial class of biconnected 2-outerplane graphs.
- For a biconnected 2-outerplane graph that satisfies the sufficient conditions, we have provided an $O(n^2)$ drawing algorithm for β-drawing the graph, for $1 < \beta < 2$.
- The specified necessary condition implies a forbidden class of biconnected 2-outerplane graphs which cannot be β-drawn, for $1 < \beta < 2$. The sufficient conditions imply a subclass of biconnected 2-outerplane graphs that are β-drawable for the same range of β values. We have shown that if a biconnected 2-outerplane graph does not belong to the forbidden class and also not to the above mentioned β-drawable class, then this graph is not necessarily β-drawable.

2 Preliminaries

In this section we first present definitions of planar and plane graphs, 1-outerplanar and 2-outerplanar graphs and then we define β-proximity graphs and state several properties of β-drawings of graphs. We assume that the reader is familiar with basic graph theoretic and graph drawing terminologies, for which we follow [West01,NR04].

A graph is *planar* if it has an embedding in the plane without any edge-crossing, except at vertices on which two or more edges are incident. A *plane graph* G is defined as a planar graph with a fixed embedding in the plane without any edge-crossing, except at vertices on which two or more edges are incident. In fact, a planar graph can have many planar embeddings and each of these embeddings is a plane graph. A plane graph divides the plane into connected regions called *faces*. The unbounded region is called the *external face*.

An *outerplanar graph* is a graph that has a planar embedding such that all the vertices lie on the external face. This graph is also known as a *1-outerplanar graph*. For a specific embedding of a graph if all the vertices are on the external face we say that the embedded graph is a *1-outerplane graph*, otherwise the embedded graph is not 1-outerplane. So the definition of 1-outerplanar graph is independent of embeddings, whereas the definition of 1-outerplane graph is concerned with a specific embedding.

These definitions can be generalized as follows. For an integer $k > 1$, an embedded graph is k*-outerplane* if the embedded graph obtained by removing all the vertices of the external face is a $(k-1)$-outerplane graph. On the other hand, we call a graph k*-outerplanar* if it has an embedding that is k-outerplane. A related notion is the *outerplanarity* of a graph which is defined as follows. For an integer $k > 0$, a graph has outerplanarity k if k is the least positive integer such that the graph is k-outerplanar.

For a biconnected 2-outerplanar graph G, let Γ be a 2-outerplane embedding of G. We call the vertices of G that are in the external face in Γ the *external vertices*. The remaining vertices are called the *internal vertices*. Each edge between two external vertices is called an *external edge*. Similarly each edge between two internal vertices is called an *internal edge*. The remaining edges each connecting an external vertex with an internal vertex are called *mixed edges*.

Let $G = (V, E)$ be a biconnected outerplanar graph. For any vertex $u \in V$, the fan of u, denoted by F_u, is the subgraph of G induced by the vertices in V that share an internal face with u in a 1-outerplanar embedding of G. Here, the vertex u is called the apex of F_u. Since G is outerplanar, F_u is also outerplanar. Let Γ be a 1-outerplanar embedding of G in which F_u has the 1-outerplanar embedding Φ. Let $u_1, u_2, \ldots, u_k$ be the vertices of neighbors of u in clockwise order in Φ. The edge (u, u_1) is called the *first edge* of F_u and (u, u_k) the *last edge* of F_u for that embedding. We call each edge (u, u_i) a *radial edge* of F_u, for $i = 2, \ldots, k-1$. Apart from the first edge, the last edge and the radial edges, all other edges of F_u are called *fan edges*. We denote by $u_{i,1}, u_{i,2}, \ldots, u_{i,m}$ the m vertices on the boundary of Φ in between u_i and u_{i+1} in clockwise order, where $1 \leq i \leq k-1$. These notations are illustrated in Fig. 2. In this paper, we adopt the notion of fan of a vertex in a biconnected 2-outerplane graph by allowing the apex of the fan to be an external vertex of the graph. A cycle C in a plane

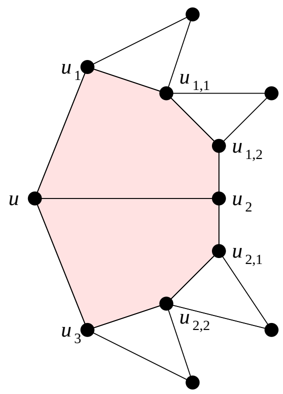

Fig. 2. Fan of a vertex u of a biconnected outerplanar graph and related notions: u is the apex; uu_1 is the first edge, uu_3 is the last edge; uu_2 is a radial edge; $u_{1,2}u_2$ and $u_2u_{2,1}$ are two fan edges and the shaded subgraph is the fan of apex u, denoted by F_u.

graph G is called a *complex cycle* if there is a vertex $v \in V(G)$ located in proper inside of C. If there are k vertices on the complex cycle C then C is called a complex k-cycle.

For any two distinct points in the plane there is an associated region parameterized by β, which is called the β-region of the two points. Kirkpatrick and Radke introduced β-regions in two variants– lune-based and circle-based β-regions [KR85]. In this paper we study only the lune-based variant. This proximity region can be further subdivided into two types– *open β-regions* (also denoted by (β)-regions) and *closed β-regions* (also denoted by $[\beta]$-regions). In the (β)-region, the boundary of the region is considered to be outside the region. However, in the $[\beta]$-region, the boundary of the region is included in the region of interest.

For two distinct points x and y in the plane the associated lune-based open β-region $R(x, y, \beta)$ and closed β-region $R[x, y, \beta]$ are defined as follows.

- For $\beta = 0$, $R(x, y, \beta)$ is the empty region and $R[x, y, \beta]$ is the straight line segment connecting x and y.
- For β in $(0, 1)$, $R(x, y, \beta)$ is the intersection of two open disks of radius $\frac{d(x,y)}{2\beta}$ passing through both x and y and $R[x, y, \beta]$ is the intersection of the two corresponding closed disks. Here, $d(x, y)$ denotes Euclidean distance between the points x and y.
- For β in $[1, \infty)$, $R(x, y, \beta)$ is the intersection of two open disks or radius $\frac{\beta d(x,y)}{2}$, centered at the points $(1 - \frac{\beta}{2})x + \frac{\beta}{2}y$ and $\frac{\beta}{2}x + (1 - \frac{\beta}{2})y$. $R[x, y, \beta]$ is the intersection of the two corresponding closed disks.

– For $\beta = \infty$, $R(x, y, \beta)$ is the open infinite strip perpendicular to the line segment xy and $R[x, y, \beta]$ is the corresponding closed infinite strip.

Examples of $R[x, y, \beta]$ are shown in Fig. 3 for several values of β. It can be observed that as the value of β increases, the corresponding β-region also increases and for $0 \leq \beta_1 < \beta_2 \leq \infty$, $R[x, y, \beta_1]$ is contained inside $R[x, y, \beta_2]$.

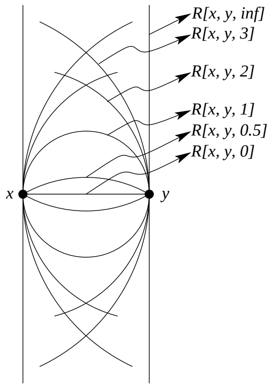

Fig. 3. $R[x, y, \beta]$ for several values of β.

Given a straight-line drawing Γ of a graph $G = (V, E)$ and the value of parameter β, we say that Γ is an *open β-drawing*, also written as *(β)-drawing*, of G if Γ maintains the following proximity constraint: $uv \in E$ if and only if $R(u, v, \beta)$ does not contain any vertex of $V - \{u, v\}$ in the drawing Γ. We can define *closed β-drawings*, written as *$[\beta]$-drawings*, similarly by considering closed β-regions.

A graph is *(β)-drawable* (or *$[\beta]$-drawable*) if it admits a (β)-drawing (or $[\beta]$-drawing). The *(β)-drawability problem* asks the question of whether an input plane graph G is (β)-drawable or not for a specified value of β. Similarly, the *$[\beta]$-drawability problem* can be defined. In this paper we simply use the notations *β-regions*, *β-drawings* or *β-drawable* graphs whenever the discussion applies for both (β)-regions and $[\beta]$-regions.

We use two angular measurements $\alpha(\beta)$ and $\gamma(\beta)$ that are defined as follows [BDLL95,BLL96,LL96].

– For $\beta \geq 0$, $\alpha(\beta) = inf\{\angle xzy \parallel z \in R[x, y, \beta], z \neq y\}$.
– For $\beta \geq 2$, $\gamma(\beta) = \angle zxy$, where $z \neq y$ is a point on the boundary of $R[x, y, \beta]$ and $d(x, y) = d(x, z)$.

The following property expresses the relationship between β and either of $\alpha(\beta)$ and $\gamma(\beta)$. The property can be proved starting from the definitions of $\alpha(\beta)$ and $\gamma(\beta)$ and using elementary geometry.

Property 1. [BDLL95,LL96]

– $\beta = sin\alpha$ for $0 \leq \beta \leq 1$ and $\frac{\pi}{2} \leq \alpha \leq \pi$.
– $\beta = \frac{1}{1-cos\alpha}$ for $1 < \beta \leq 2$ and $0 \leq \alpha < \frac{\pi}{2}$.
– $\beta = \frac{1}{cos\gamma}$ for $2 \leq \beta \leq \infty$ and $\frac{\pi}{3} \leq \gamma < \frac{\pi}{2}$.

3 β-Drawability of Biconnected 2-Outerplane Graphs

In this section we give characterization of biconnected 2-outerplane graphs for having a β-drawing. We provide a constructive proof of our claim which gives an $O(n^2)$ algorithm for β-drawing a biconnected 2-outerplane graph satisfying a set of sufficient conditions.

There are biconnected 2-outerplane graphs that are not β-drawable for $\beta \in (1, 2)$. For example, let us consider a biconnected 2-outerplane graph with exactly 4 external vertices and exactly 1 internal vertex. Suppose that we want to achieve a [1]-drawing of this graph. No matter where we place the four external vertices, the internal vertex will be inside the proximity region of at least one of the four pairs of neighboring external vertices. And hence the graph is not [1]-drawable. This geometric property has been outlined in [LL96]. The mentioned graph will not be β-drawable for $\beta > 1$ as well, since the proximity regions will only increase and we will have no place to position the internal vertex. The same situation will arise for a 2-outerplane graph with three external vertices. Thus we can arrive at the following lemma.

Lemma 1. *Let G be a biconnected 2-outerplane graph. Then G has no β-drawing for $1 < \beta < 2$ if G has less than five external vertices.*

By definition, Lemma 1 implies that some of graphs in the class of biconnected 2-outerplane graphs are not β-drawable for $\beta \in (1, 2)$. We are now interested in finding a subclass of biconnected 2-outerplane graphs that are β-drawable for $1 < \beta < 2$. The following theorem characterizes a subclass of biconnected 2-outerplane graphs that are β-drawable for $1 < \beta < 2$.

Theorem 1. *A biconnected 2-outerplane graph G is β-drawable for $\beta \in (1, 2)$ if G satisfies the following conditions:*

1. *There are at least five external vertices; and*
2. *There is an external vertex u such that the fan F_u has all of the following properties:*
   (a) *F_u is biconnected 1-outerplane;*
   (b) *F_u contains all the internal vertices; and*
   (c) *Every vertex in F_u has at most one neighbor outside F_u and every vertex outside F_u has at most one neighbor in F_u.*

In the rest of this section we provide a constructive proof of Theorem 1. But before going on to the proof, an interesting question arises regarding a biconnected 2-outerplane graph G that satisfies all the conditions in Theorem 1: is there any possibility of the occurrence of a complex 3-cycle or a complex 4-cycle in G? The following lemma, whose proof is omitted in this extended abstract, confirms that this can never happen.

Lemma 2. *Let G be a biconnected 2-outerplane graphs satisfying the conditions in Theorem 1. Then every 3-cycle, as well as every 4-cycle, of G is a face.*

We are now going to present a constructive proof of Theorem 1. The outline of the proof is as follows. Let G be a biconnected 2-outerplane graph satisfying the conditions specified in Theorem 1. According to the conditions, G has at least five external vertices. In addition to that, G has an external vertex u such that the fan F_u satisfies the Conditions 2(a), 2(b) and 2(c). We first have to find this external vertex u. Once such an external vertex u has been found, then we draw the fan F_u. We next draw the remaining graph $G-(V(F_u))$ and add edges between the vertices of F_u and the vertices of $G-(V(F_u))$. We finally prove the correctness of the drawing procedure.

3.1 Finding an appropriate apex

In the next lemma we give an algorithm that takes as input a biconnected 2-outerplane graph G that satisfies the conditions stated in Theorem 1 and finds a set of external vertices of G such that for each vertex u in this set, the corresponding fan F_u satisfies Conditions 2(a), 2(b) and 2(c) of Theorem 1. We call such a set of external vertex the *set of candidate apices*.

Lemma 3. *Let G be a biconnected 2-outerplane graph satisfying all the conditions specified in Theorem 1. Then the set C of external vertices u for which F_u satisfies Conditions 2(a), 2(b) and 2(c) of Theorem 1 can be found in $O(n^2)$ time, where n is the number of vertices of G.*

Outline of proof. For each external vertex u, it can be checked whether the fan F_u satisfies all the required conditions: whether a vertex of the fan at most one neighbor outside the fan, whether a vertex outside the fan has at most one neighbor in the fan and whether the fan is biconnected 1-outerplane. For a specific fan F_u, checking these three conditions requires $O(n)$ time. Since these three conditions are checked for fans of all the external vertices of the graph, the total time complexity of this algorithm is $O(n^2)$. $\mathcal{Q.E.D.}$

3.2 Drawing the fan

We can obtain a set C of candidate apices using Lemma 3. We can choose any apex from this set for the purpose of β-drawing of the graph. In the next lemma we show how we can β-draw the fan F_u corresponding to an apex $u \in C$. This lemma is due to Lenhart and Liotta [LL96]. We change Lemma 4 slightly from the original lemma given in [LL96] to fit our purpose.

Lemma 4. *Let F_u be a biconnected outerplane fan with apex u. Then F_u can be β-drawn inside a triangle Δabc, for $\beta \in (1, 2)$, $\angle abc > \frac{\pi}{2}$ and $\angle bac < \frac{\pi}{4}$. Furthermore, this drawing has the property that the fan edges form a convex chain such that for any three vertices v_1, v_2 and v_3 on the chain in clockwise order, $\angle v_1 v_2 v_3 > \frac{\pi}{2}$.*

Outline of proof. The statement can be proved constructively by induction on the number of neighbors of u, as shown in [LL96].

$\mathcal{Q.E.D.}$

3.3 Drawing the remaining graph

Once we find an apex u by Lemma 3, we can β-draw the fan F_u that satisfies the conditions in Theorem 1 according to Lemma 4. Suppose that the fan F_u has been drawn inside an obtuse triangle Δabc such that $\angle abc > \frac{\pi}{2}$ and $\angle bac < \frac{\pi}{4}$. We now show how we can draw the remaining part of the graph so that the graph G, as a whole, is correctly β-drawn.

Regions for drawing F_u and $G - V(F_u)$

First of all, we want to find two regions $\mathcal{R}_1$ and $\mathcal{R}_2$ in the plane such that for any two points x and y in $\mathcal{R}_1$, $R[x, y, \beta]$ never overlaps $\mathcal{R}_2$ and vice versa for any two points in $\mathcal{R}_2$. The intention is to place the drawing of F_u in $\mathcal{R}_1$ and the drawing of $G - V(F_u)$ in $\mathcal{R}_2$. Then edges will be added in between vertices of F_u and vertices of $G - V(F_u)$ so that the proximity constraints are not violated. Of course, there are other issues apart from selection of such regions, which will be considered as well.

Let us compute two non-parallel and non-perpendicular straight lines L_1 and L_2 such that the acute angle δ at the intersection point of the two straight lines satisfies $\frac{\pi}{4} < \delta < \alpha(\beta)$. The two intersecting straight lines divide the plane into four regions. Among these four regions two regions contain the acute angle δ and these two are the regions of our interest: $\mathcal{R}_1$ and $\mathcal{R}_2$. The constraint $\delta < \alpha(\beta)$ ensures that the proximity region of any two points of $\mathcal{R}_1$ is outside the region $\mathcal{R}_2$ and vice versa.

We place the triangle Δabc, inside which F_u is drawn, on the plane as follows (please see Fig. 4 for illustration):

– All the vertices of F_u are inside the region $\mathcal{R}_2$ and with respect to the convex chain of fan vertices, u is positioned opposite to the region $\mathcal{R}_1$.
– Let uu_1 be the first edge and uu_k be the last edge of F_u, for $k \geq 2$. The line segments uu_1 and uu_k, when extended, intersect the lines L_1 and L_2 at points p and q, respectively, on the boundary of the region $\mathcal{R}_1$. Since the acute angle between the lines uu_1 and uu_k is less than $\frac{\pi}{4}$, we need to impose the constraint $\delta > \frac{\pi}{4}$ so that the above mentioned intersections are always possible.

Placement of the vertices of $G - V(F_u)$

The fan F_u being drawn in the region $\mathcal{R}_2$, the vertices of $G - V(F_u)$ are drawn in the region $\mathcal{R}_1$ as follows:

First, compute an arc xzy in the region $\mathcal{R}_1$ with a as the center and x and y being the intersection points with the extended line segments ap and aq respectively, as illustrated in Fig. 4. Suppose that ut_1 is the first edge and ut_m

is the last edge of the fan F_u and the chain of fan vertices in clockwise order are: $t_1, t_2, \ldots, t_m$. Let us now compute the rays emanating from the point u, passing through the chain of fan vertices $t_1, t_2, \ldots, t_m$ and intersecting the arc xzy at the points $w_1, w_2, \ldots, w_m$ respectively. See Fig. 4 for illustration.

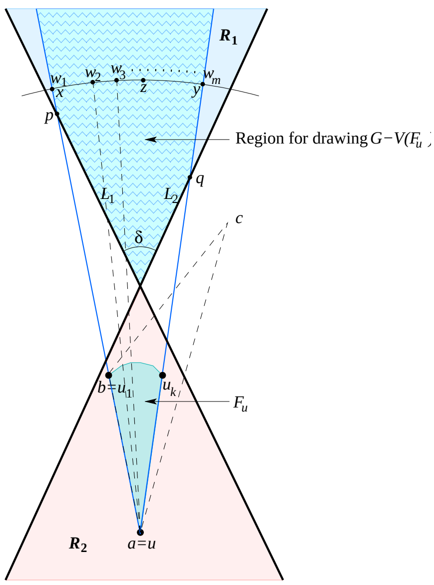

Fig. 4. Illustration of the drawing procedure.

Vertices of $G - V(F_u)$ can be divided into two types–

- Type 1: The vertices of $G - V(F_u)$ that share an internal face with the vertices of F_u
  The vertices of Type 1 can further be subdivided into two subtypes:
  - Type 1A: Some of the vertices of Type 1 are adjacent to the vertices of F_u. We name these vertices Type 1A.
  - Type 1B: The remaining vertices of Type 1 are nonadjacent to any vertex of F_u. We name these vertices Type 1B.
- Type 2: The vertices of $G - V(F_u)$ that do not share an internal face with any vertex of F_u

Placement of vertices of Type 1A:

Note that by the property of the graph G, for each of the fan vertices t_i, for $1 \le i \le m$, there can be at most one neighbor in $G - V(F_u)$. For $1 \le i \le m$, if $\exists v \in V(G - V(F_u))$ such that $v \leftrightarrow t_i$ then place the vertex v on the arc xzy at the point w_i that has been computed previously. Let the newly placed vertices be $v_1, v_2, \ldots, v_l$ in clockwise order, where $l \le m$ since some of the m fan vertices might not have neighbors in $G - V(F_u)$.

Placement of vertices of Type 1B:

As mentioned above, the Type 1B are those vertices of Type 1 that are not adjacent to any vertex of F_u, but share an internal face with a vertex of F_u. We can find these vertices by clockwise traversal of the external face of G. For $1 \le i \le l-1$, the vertices on the external face in between v_i and v_{i+1}, such that v_i and v_{i+1} are nonadjacent, form the set of the vertices of Type 1B. We place these vertices of Type 1B on a convex curve maintaining the relative order by induction as follows. We prove the correctness of the placement later on. See Fig. 5 for illustration.

- *Base case*
  Suppose that there are exactly two vertices of Type 1A: v_i and v_{i+1}, where $v_i \leftrightarrow t_j$, $v_{i+1} \leftrightarrow t_{j'}$ and $t_j, t_{j'} \in V(F_u)$. Suppose that $v_{i,1}$, $v_{i,2}$, $\ldots$, $v_{i,r}$ are vertices of Type 1B from v_i to v_{i+1} in clockwise order and $t_{j,1}$, $t_{j,2}$, $\ldots$, $t_{j,s}$ are fan vertices of F_u also in clockwise order from t_j to $t_{j'}$.
  Let gg' and $g_1{g_1}'$ be the tangents to the arc xzy at the points v_i and v_{i+1} respectively. From the proof of Lemma 4, $\angle ut_jt_{j,1} < \frac{\pi}{2}$. So, $\angle t_{j,1}t_jv_i > \frac{\pi}{2}$. Let us draw a straight line t_jd such that $\angle t_{j,1}t_jd \ge \frac{\pi}{2}$ and gg' intersects the line t_jd at the point d' "below" the point d'' at which $g_1{g_1}'$ intersects t_jd. Note that intersections in this way are always possible since $\angle t_{j,1}t_jv_i > \frac{\pi}{2}$. Furthermore, the line t_jd will intersect the arc xzy at a point in between v_i and v_{i+1}, because of the convex placement of the fan vertices and $\angle t_{j,1}t_jd \ge \frac{\pi}{2}$ and $\angle t_{j,s}t_{j'}v_{i+1} > \frac{\pi}{2}$. Here, the word "below" indicates relative order of the intersection points on the line t_jd with respect to u.
  Let w be a point on the arc xzy such that w, v_i, v_{i+1} occur in clockwise order and the straight line wv_iw' intersects the line t_jd at the point d''' in between d' and d'' such that $\angle v_id'''v_{i+1} > \frac{\pi}{2}$. Let hh' be perpendicular to the line ww' at the point v_i. Let us draw an arc $x_1v_iy_1$ with center on the straight line hh' and y_1 being a point on the line t_jd in between d''' and d'. Note that this is always possible since $wd'''w'$ is the tangent to the arc at the point v'. We place the vertices of Type 1B $v_{i,1}$, $v_{i,2}$, $\ldots$, $v_{i,r}$ on the arc $x_1v_iy_1$ in between the points v_i and y_1 maintaining their relative order. This placement guarantees that for any three vertices of Type 1 s_1, s_2 and s_3 in clockwise order around the external face of G, $\frac{\pi}{2} < \angle s_1s_2s_3 < \pi$.
- *Induction Step*
  Suppose that if there are less than k vertices of Type 1A, for some positive integer k, then the vertices of Type 1B can be placed with the property that for any three consecutive vertices of Type 1 v_1, v_2 and v_3, $\frac{\pi}{2} < \angle v_1v_2v_3 < \pi$. Now assume that there are $k \ge 2$ vertices of Type 1A with v_i, v_{i+1} and v_{i+2} being the last three vertices of Type 1A in clockwise traversal of the external face of G. By the induction hypothesis we can place all the vertices of Type 1B occurring before v_{i+1} in clockwise order maintaining the required constraint. We now have to correctly place the vertices of Type 1B in between v_{i+1} and v_{i+2}.
  Let $v_{i+1} \leftrightarrow t_{j'}$ and $v_{i+2} \leftrightarrow t_{j''}$, where $t_{j'}, t_{j''} \in V(F_u)$ and $t_{j',1}$, $t_{j',2}$, $\ldots$, $t_{j',s'}$ are the fan vertices of F_u from $t_{j'}$ to $t_{j''}$ in clockwise order. Let $h_1{h_1}'$

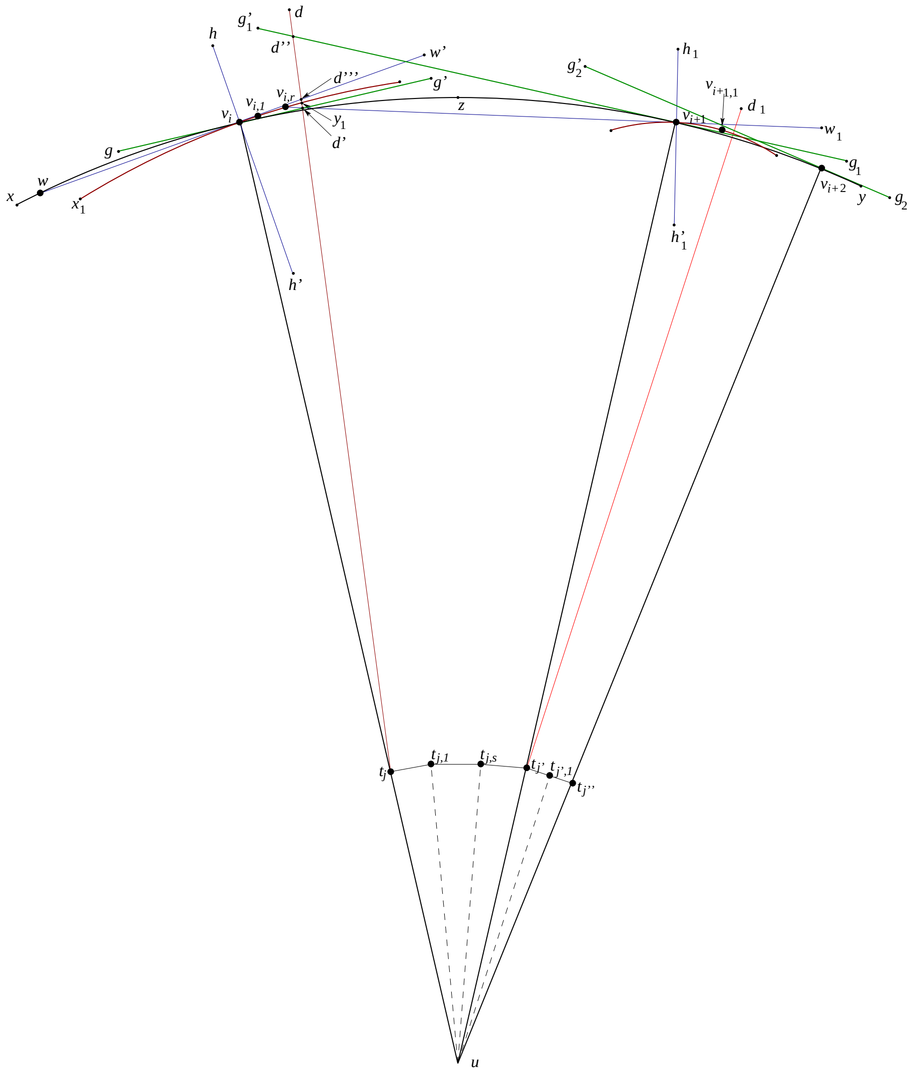

Fig. 5. Placement of vertices of Type 1B.

be perpendicular to the line $v_{i,r}v_{i+1}$ at the point v_{i+1}. Suppose that g_2g_2' be the tangent to the arc xzy at the point v_{i+2}. By Lemma 4, $\angle t_{j',1}t_{j'}u < \frac{\pi}{2}$. Let us compute a straight line $t_{j'}d_1$ such that $\angle t_{j',1}t_{j'}d_1 \geq \frac{\pi}{2}$ and $t_{j'}d_1$ intersects g_1g_1' at a point "below" its intersection points with g_2g_2' and with $v_{i,r}v_{i+1}w_1$. Note that this is always possible. Furthermore, using the same argument as in the base case, it can be shown that the line $t_{j'}d_1$ intersects the arc xzy at a point between v_{i+1} and v_{i+2}.

Now, let us compute an arc with center on the line $h_1{h_1}'$ and passing through v_{i+1} and a point y_2 on the line $t_{j'}d_1$ such that y_2 is "above" the intersection point of the lines $t_{j'}d_1$ and $g_1{g_1}'$, "below" the intersection point of $t_{j'}d_1$ and $g_2{g_2}'$ and also "below" the intersection point of $t_{j'}d_1$ and $v_{i,r}v_{i+1}w_1$. Here, the words "above" and "below" indicate relative order of the intersection points on the line $t_{j'}d_1$ with respect to u.
It can be proved that all the vertices of Type 1 have been placed on a convex curve such that for any three vertices of Type 1 s_1, s_2 and s_3 in clockwise order, $\angle s_1s_2s_3 > \frac{\pi}{2}$.

Placement of the vertices of Type 2:

Since all the internal vertices of G are in the fan F_u, none of the vertices of Type 2 can be adjacent to more than two vertices of Type 1, otherwise some of the vertices of Type 1 become internal vertices of G. We have placed the vertices of Type 1 so that the chain of edges formed by the vertices of Type 1 is convex. Let e_1 be an edge between two vertices of Type 1 w_1 and w_2 and a subset of the vertices of Type 2 shares an internal face with the vertex w_1. By Lemma 4, we can draw this subset of vertices inside an obtuse triangle $\Delta w_1w_2p_1$ with $\angle w_1w_2p_1 > \frac{\pi}{2}$. Similarly if e_2 is the next edge with w_2 and w_3, which are vertices of Type 1, as endpoints, then the subset of the vertices of Type 2 sharing an internal face with the vertex w_2 can be drawn inside an obtuse triangle $\Delta w_2w_3p_2$, with $\angle w_2w_3p_2 > \frac{\pi}{2}$. Here, we can choose the points p_1 and p_2 such that $\angle w_1w_2p_1 > \frac{\pi}{2}$ and $\angle p_1w_2p_2 > \frac{\pi}{2}$, since every external angle on the chain of the vertices of Type 1 is greater than π because of convexity. We perform the placement of the vertices of Type 2 recursively until all of them have been placed.

3.4 Proof of proximity constraints

It can be proved that the drawing procedure described above maintains the proximity constraints. Proving that the proximity constrained are satisfied in the following cases is sufficient for this purpose– any two vertices of F_u, any two vertices of Type 1, any two vertices of Type 2, a vertex of F_u and another of Type 1, a vertex of F_u and another of Type 2, a vertex of Type 1 and another of Type 2.

4 Forbidden 2-Outerplane Graphs

Up to this point, we have found a set of sufficient conditions and proved that if a biconnected 2-outerplane graph G satisfies these conditions then G can be β-drawn, for $1 < \beta < 2$. But what can we say about the biconnected 2-outerplane graphs that do not satisfy these conditions? Are these graphs β-drawable? In this section we address this question. We define a class of graphs as forbidden if no graph of this class is β-drawable for a specified value of β.

Lemma 1 specifies a necessary condition for β-drawability of biconnected 2-outerplane graphs. The necessary condition is that if G is a biconnected 2-outerplane graph having less than five external vertices then G has no β-drawing, for $1 < \beta < 2$. Thus we have identified a forbidden class of biconnected 2-outerplane graphs, that is the class of biconnected 2-outerplane graphs having less than five external vertices. On the other hand, by Theorem 1 we get a subclass of biconnected 2-outerplane graphs that is β-drawable for $1 < \beta < 2$.

We now show that the biconnected 2-outerplane graphs violating the sufficient conditions specified in Theorem 1 are not always β-drawable for $1 < \beta < 2$.

Theorem 2. *Let $\mathcal{F}$ be the class of biconnected 2-outerplane graphs that do not satisfy at least one of the conditions specified in Theorem 1. Then every graph in the class $\mathcal{F}$ is not necessarily β-drawable, for* $1 < \beta < 2$.

Outline of proof. For each of the sufficient conditions, we can find a member of $\mathcal{F}$ that does not satisfy the condition and that is not β-drawable for $1 < \beta < 2$. In proving this claim we use Lemma 1.

$\mathcal{Q.E.D.}$

5 Conclusion

In this paper we have studied the β-drawability problem for biconnected 2-outerplane graphs. We have specified a necessary condition for β-drawability of biconnected 2-outerplane graphs, for $1 < \beta < 2$. We have found sufficient conditions for β-drawability of biconnected 2-outerplane graphs, for $1 < \beta < 2$. For a biconnected 2-outerplane graph that satisfies the sufficient conditions, we have given a drawing algorithm for β-drawing the graph, for $1 < \beta < 2$. Finally, we have identified a forbidden class of biconnected 2-outerplane graphs and proved that a biconnected 2-outerplane graph that does not satisfy the mentioned sufficient conditions, is not necessarily β-drawable. We conclude this paper with the following open problems–

- Complete characterization of biconnected 2-outerplane graphs is yet to be done. In this problem, one has to provide necessary and sufficient conditions for β-drawability of a biconnected 2-outerplane graph. Interestingly, complete characterizations of trees and outerplanar graphs are still open problems as well [BDLL95,LL96].
- β-drawings of trees, outerplanar graphs and biconnected 2-outerplane graphs in polynomial area is an open problem. Interestingly, the simplest of these problems, i.e. polynomial area β-drawings of trees for $\beta = 1$, is still unsolved.
- Characterization of β-drawability of k-outerplane graphs, for any $k \geq 1$, will be a very interesting finding.

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