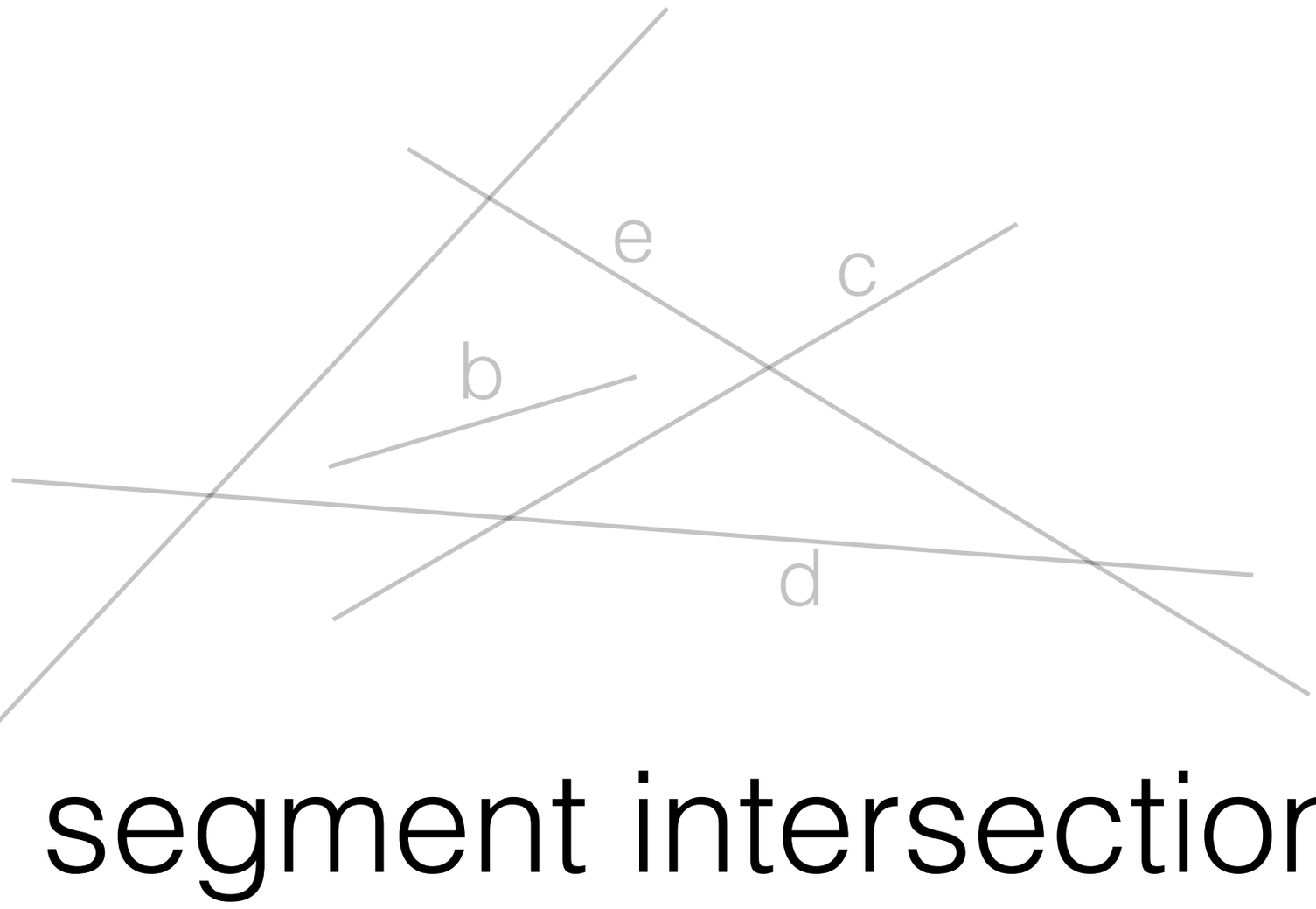


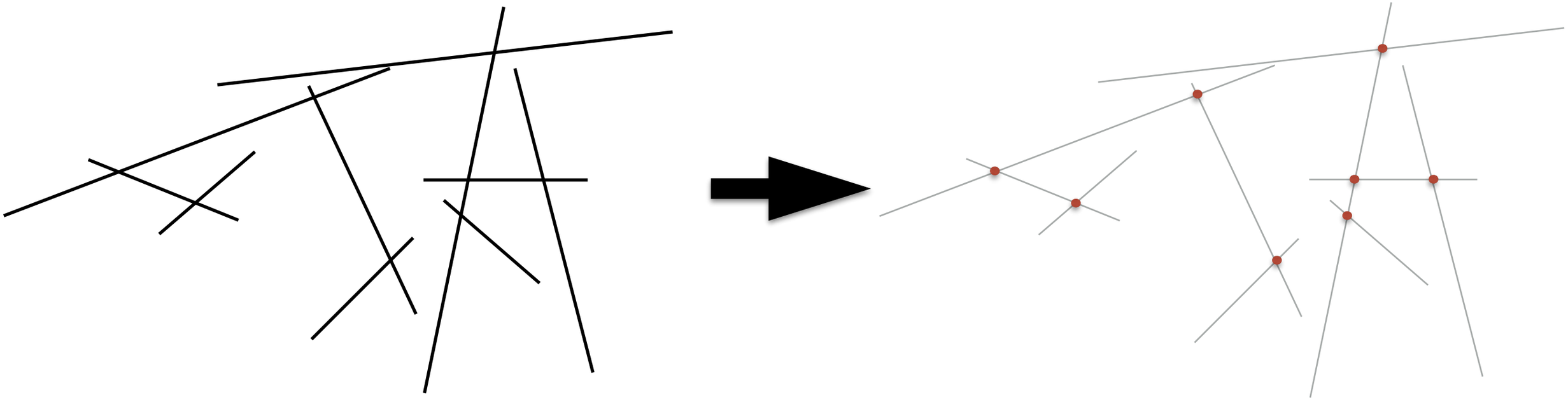

# Line segment intersection

Computational Geometry
[csci 3250]

Laura Toma
Bowdoin College

# Line segment intersection

Problem: Given a set of line segments in 2D, find all their pairwise intersections.



# Line segment intersection

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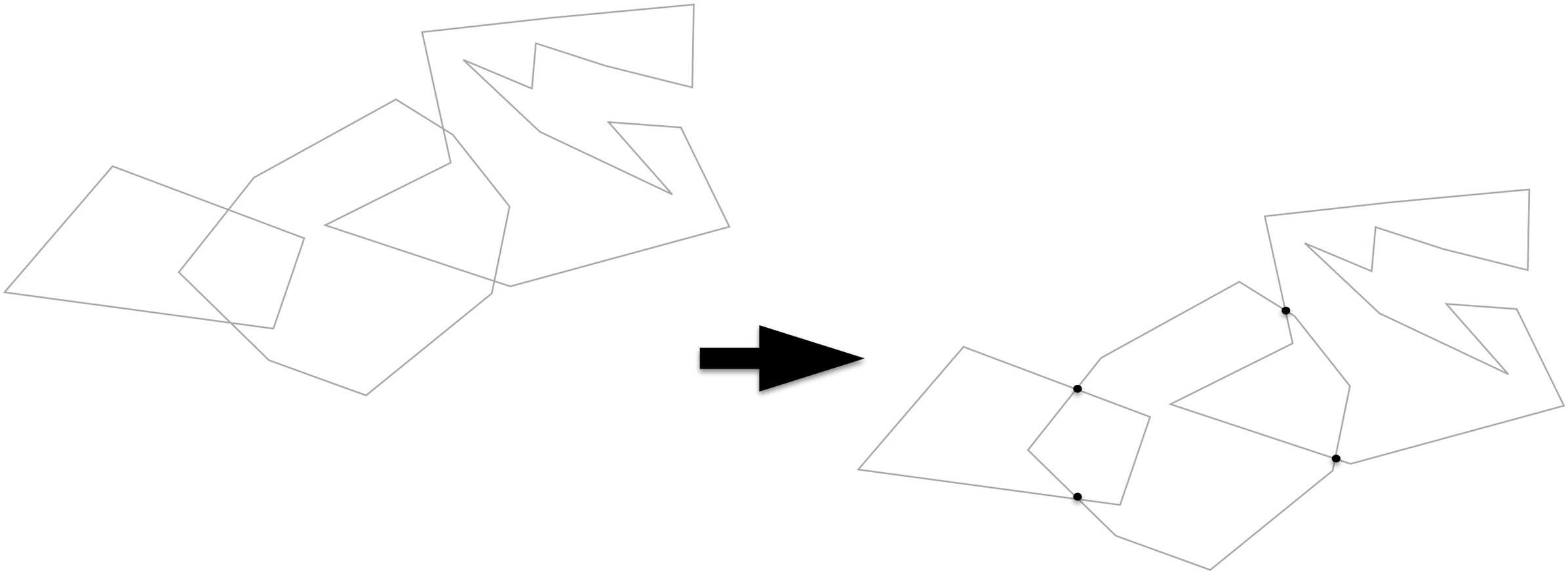

# Line segment intersection

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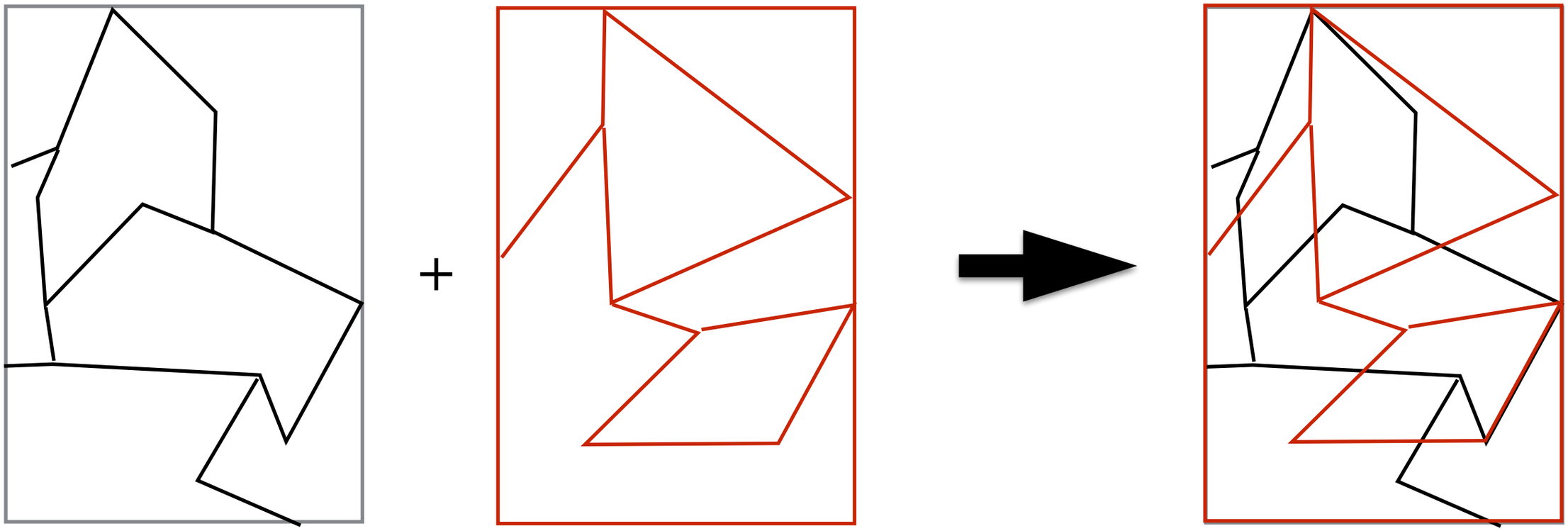
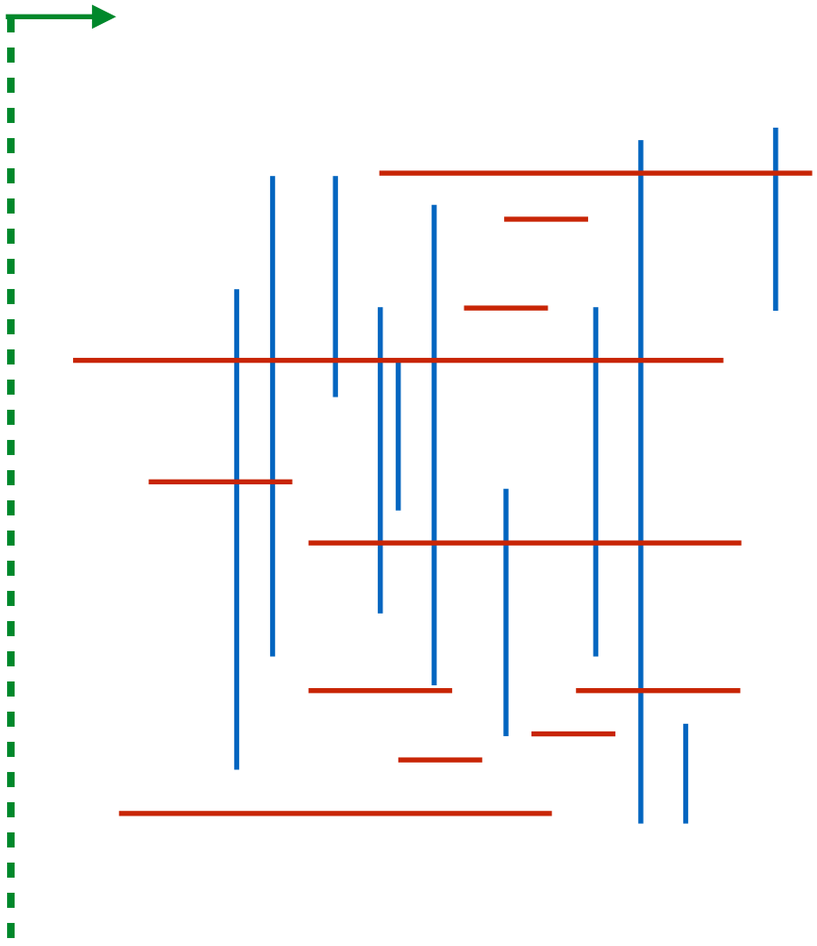
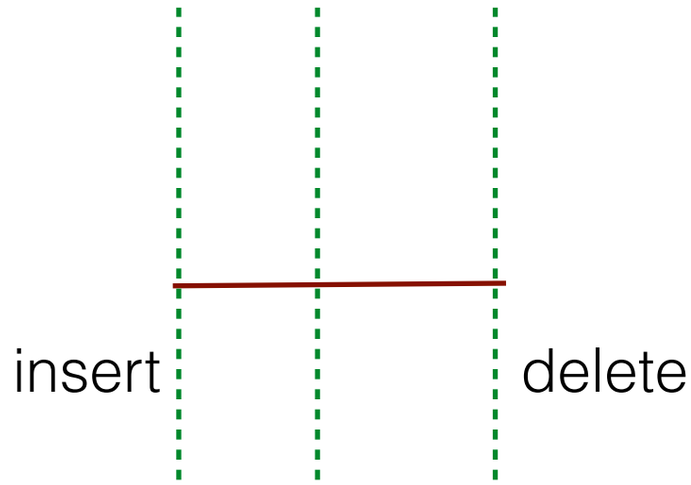


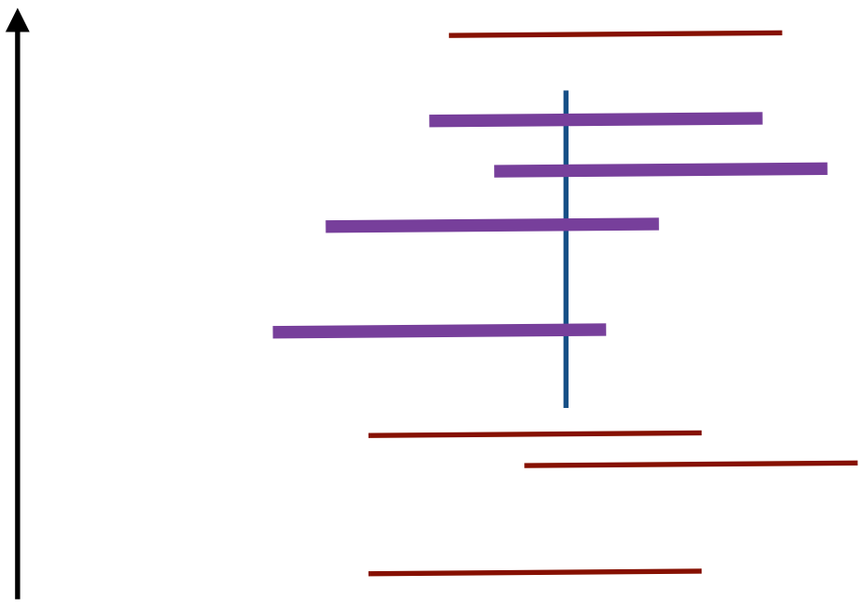
# Orthogonal line segment intersection



line sweep technique

solve the problem behind the line



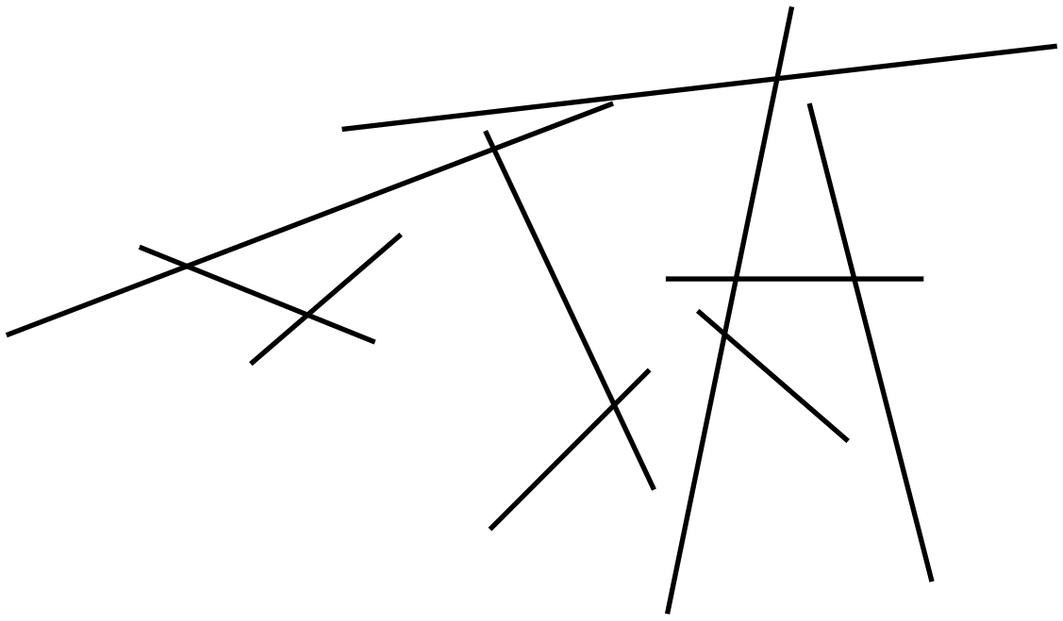



AS: segments in order of y

Result: The intersections of a set of n orthogonal segments in the plane can be found in O(n lg n + k) time.

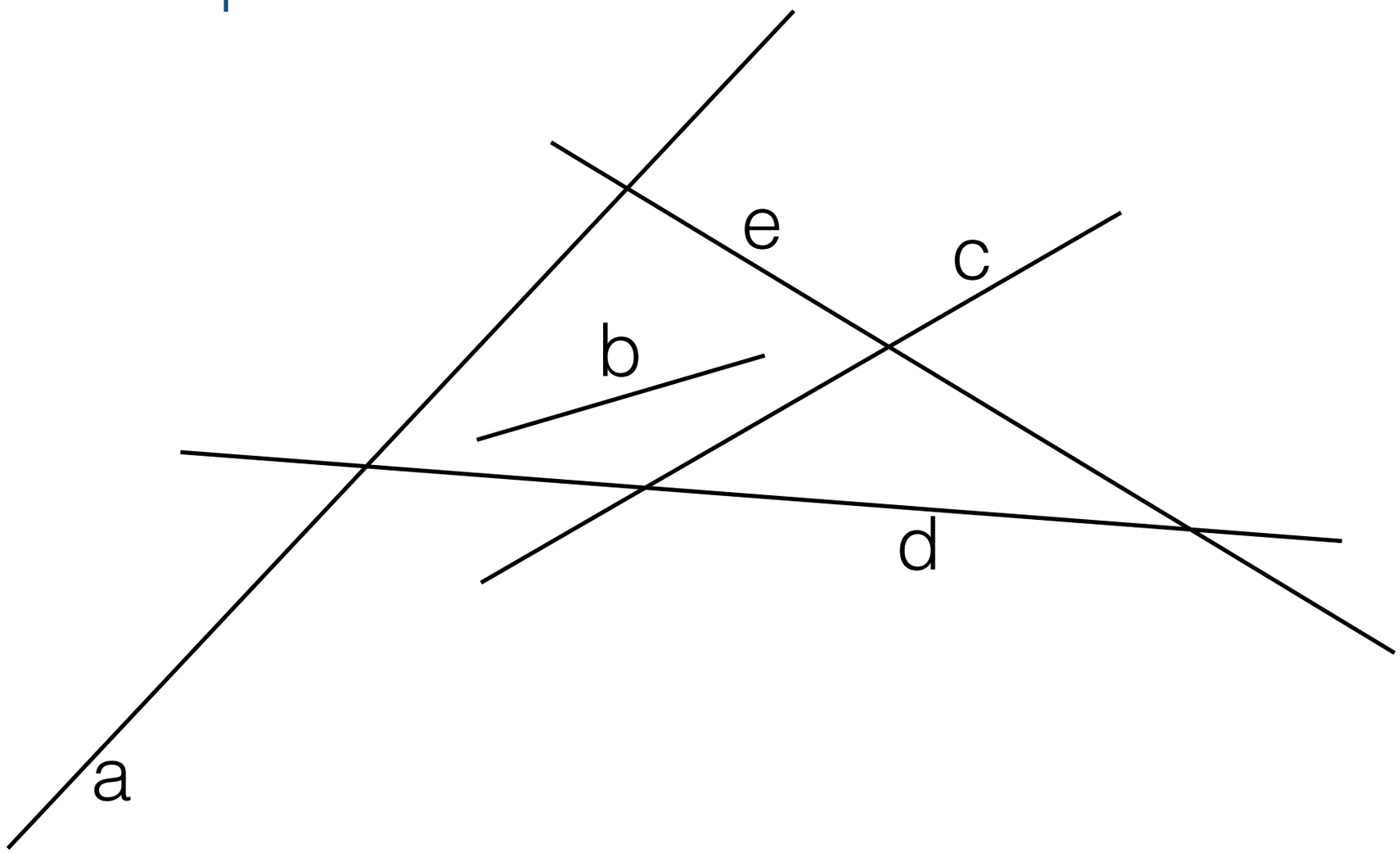
# General line segment intersection

- n: size of the input (number of segments)
- k: size of output (number of intersections)

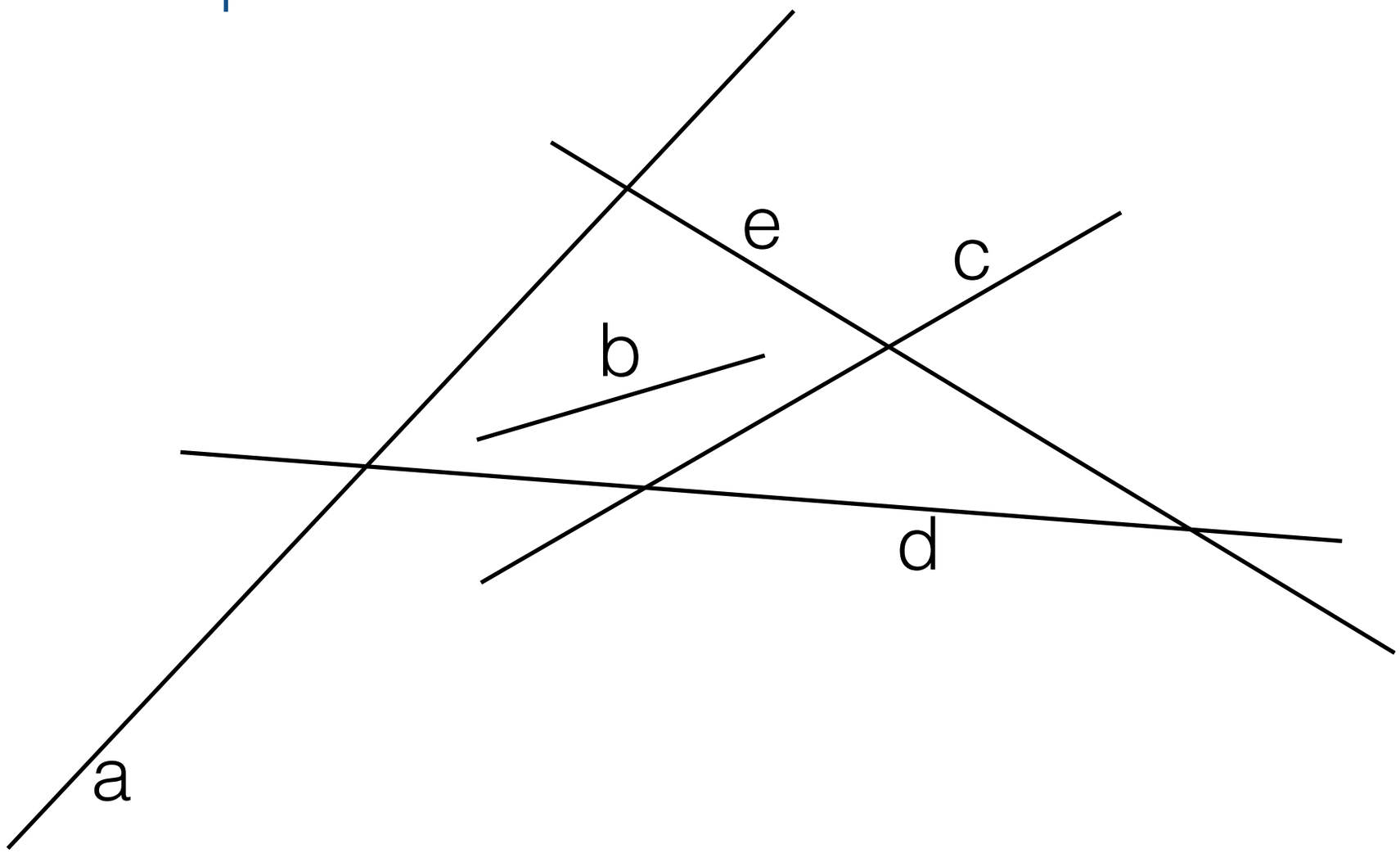


- Extend sweep line idea
- We'll get an overall bound of O(n lg n + k lg n) which improves on the naive $O(n^2)$ when k is small
- The algorithm was developed by Jon Bentley and Thomas Ottman in 1979
- Simple (in retrospect), elegant and practical

# The sweep

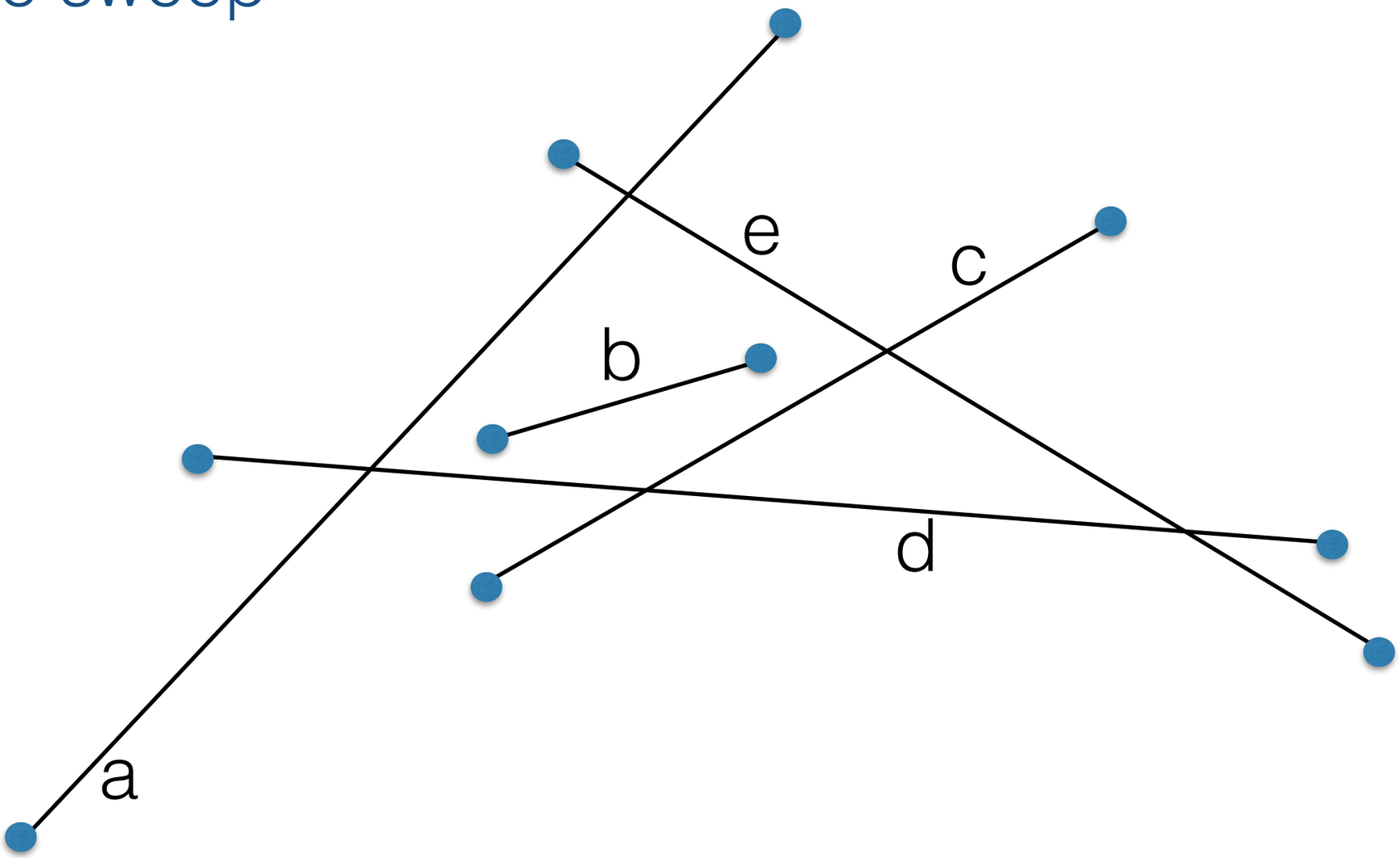


# The sweep




- Let X be the set of all x-coords of segments

# The sweep

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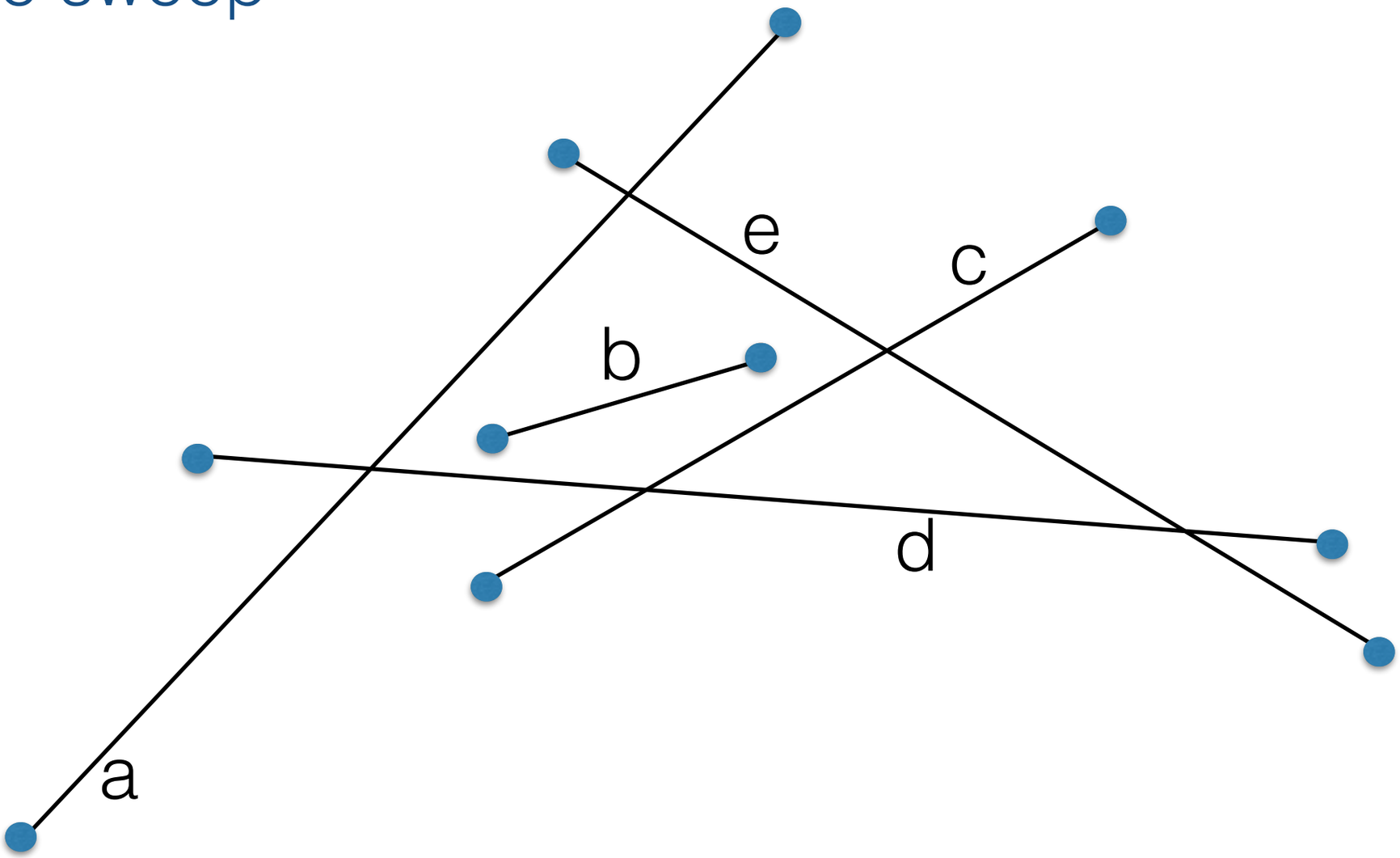
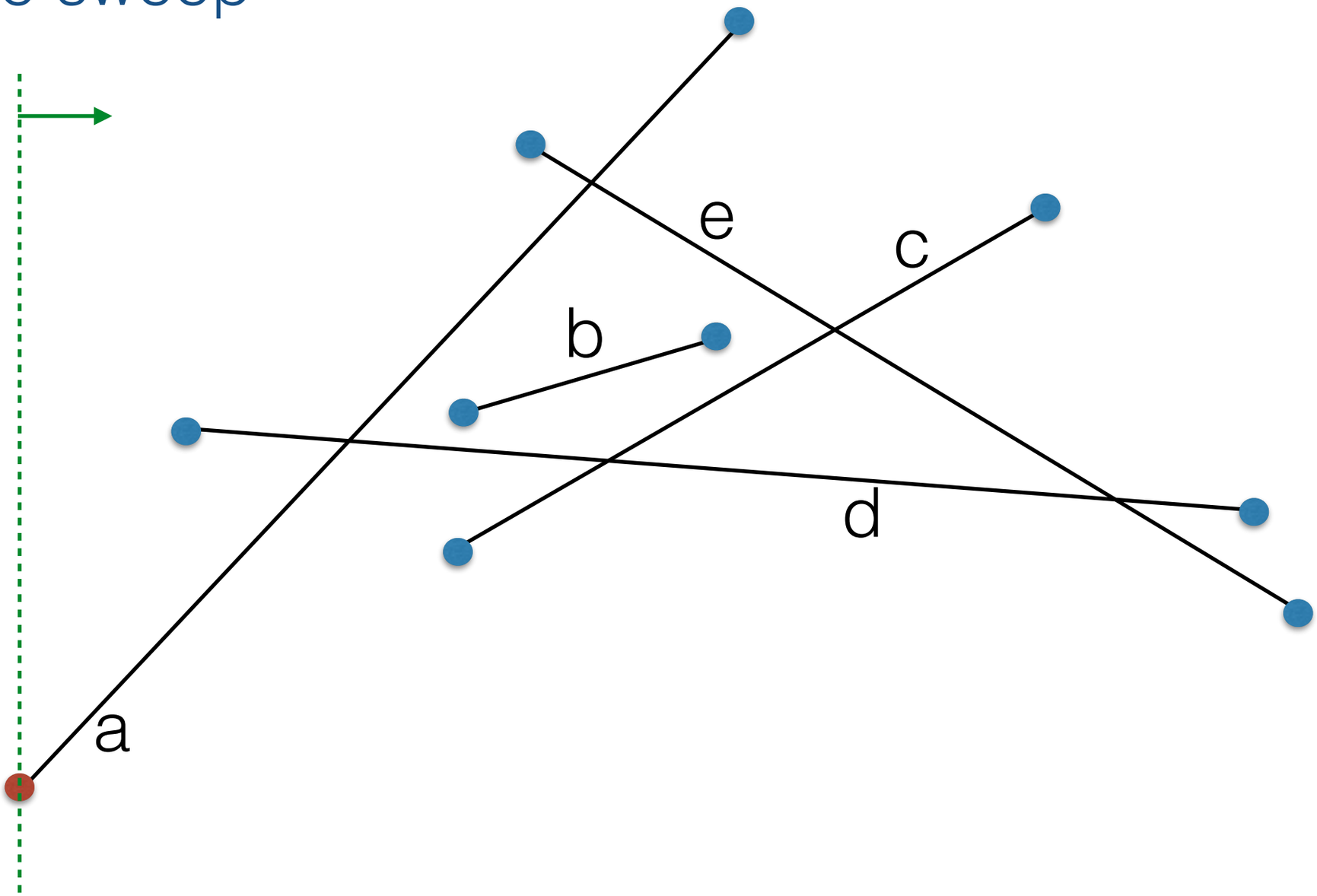



- Let X be the set of all x-coords of segments
- Traverse the events in X in order

# The sweep

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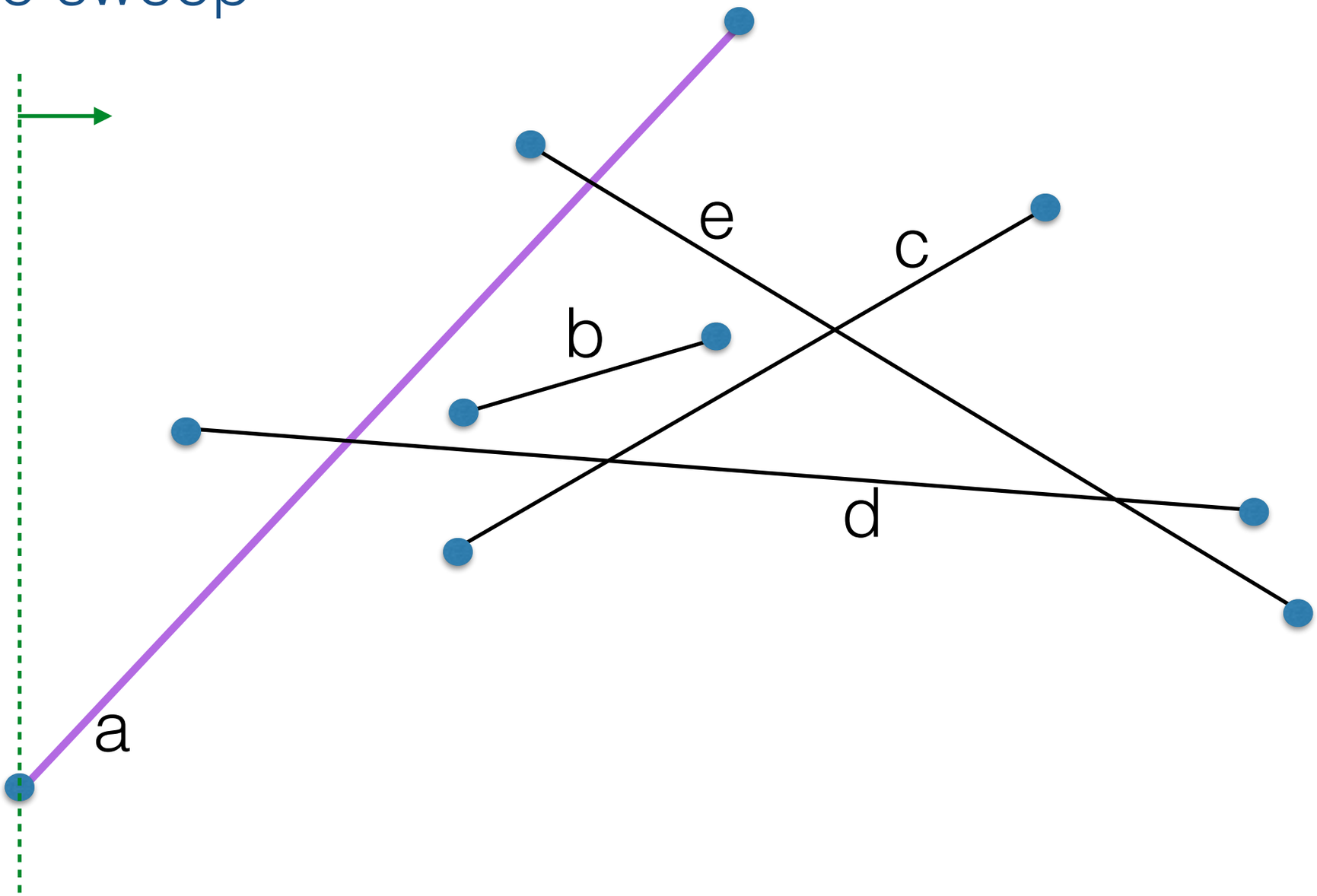


- Let X be the set of all x-coords of segments
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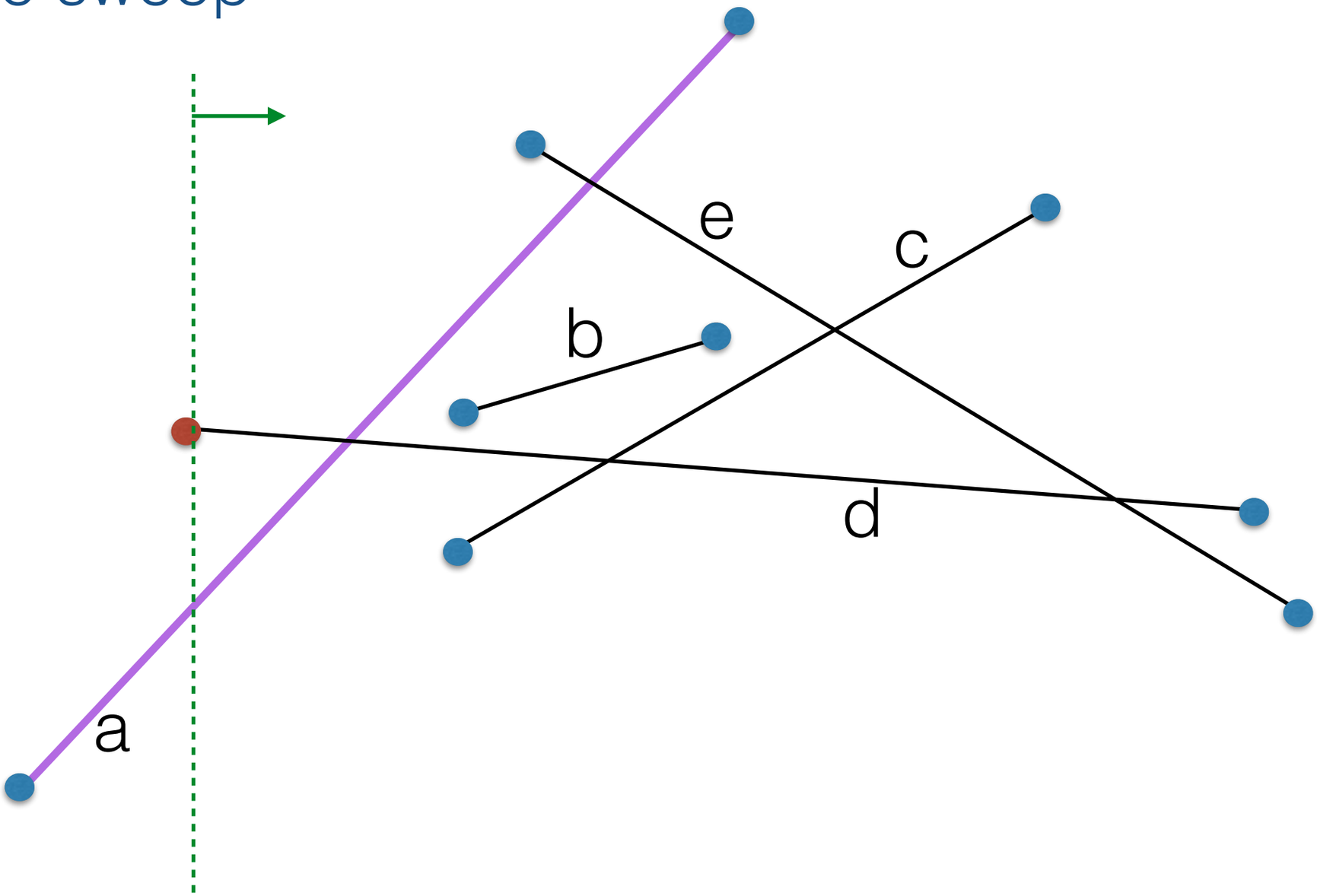
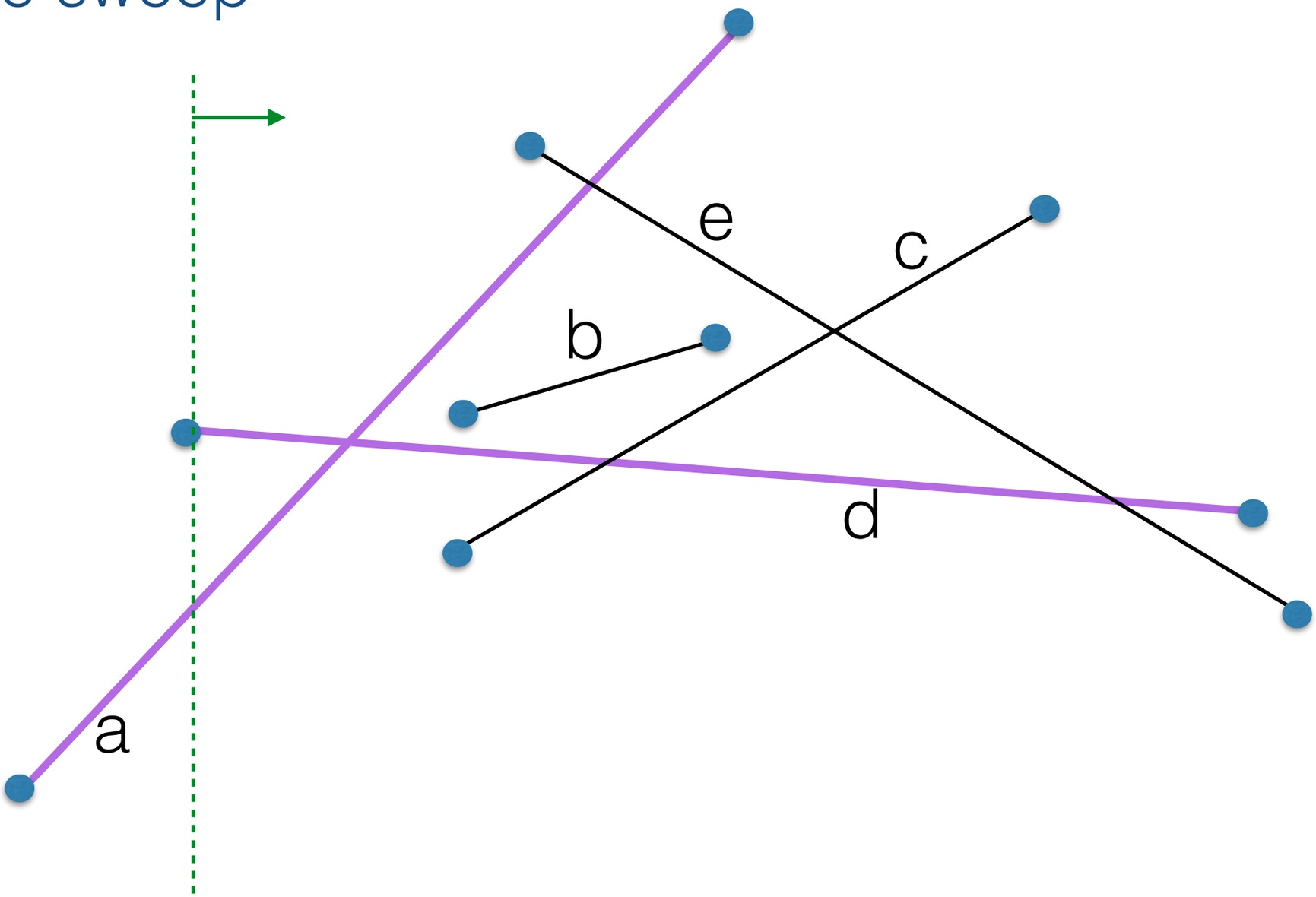
# The sweep




- Let X be the set of all x-coords of segments
- Traverse the events in X in order

# The sweep




- Let X be the set of all x-coords of segments
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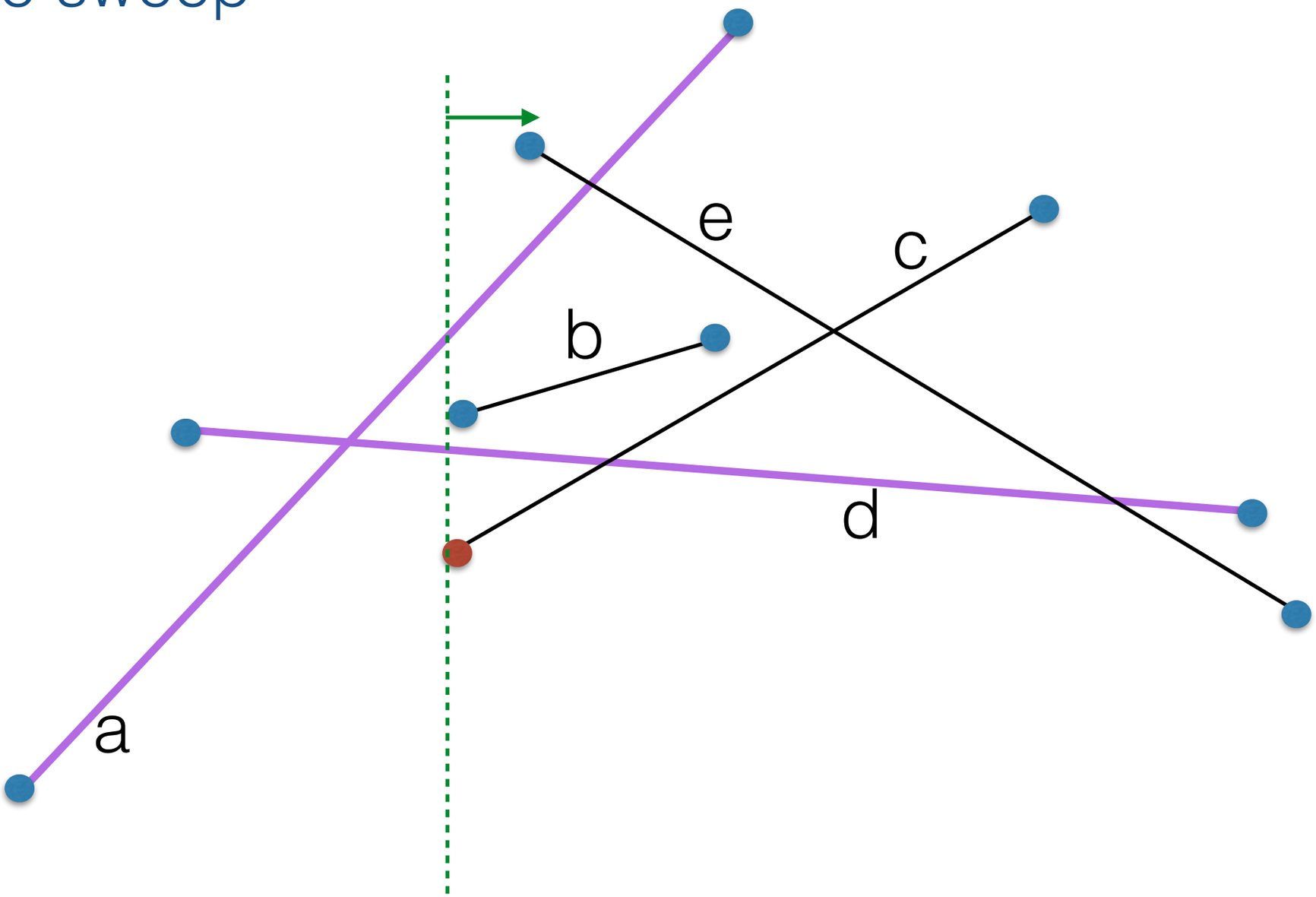
# The sweep




- Let X be the set of all x-coords of segments
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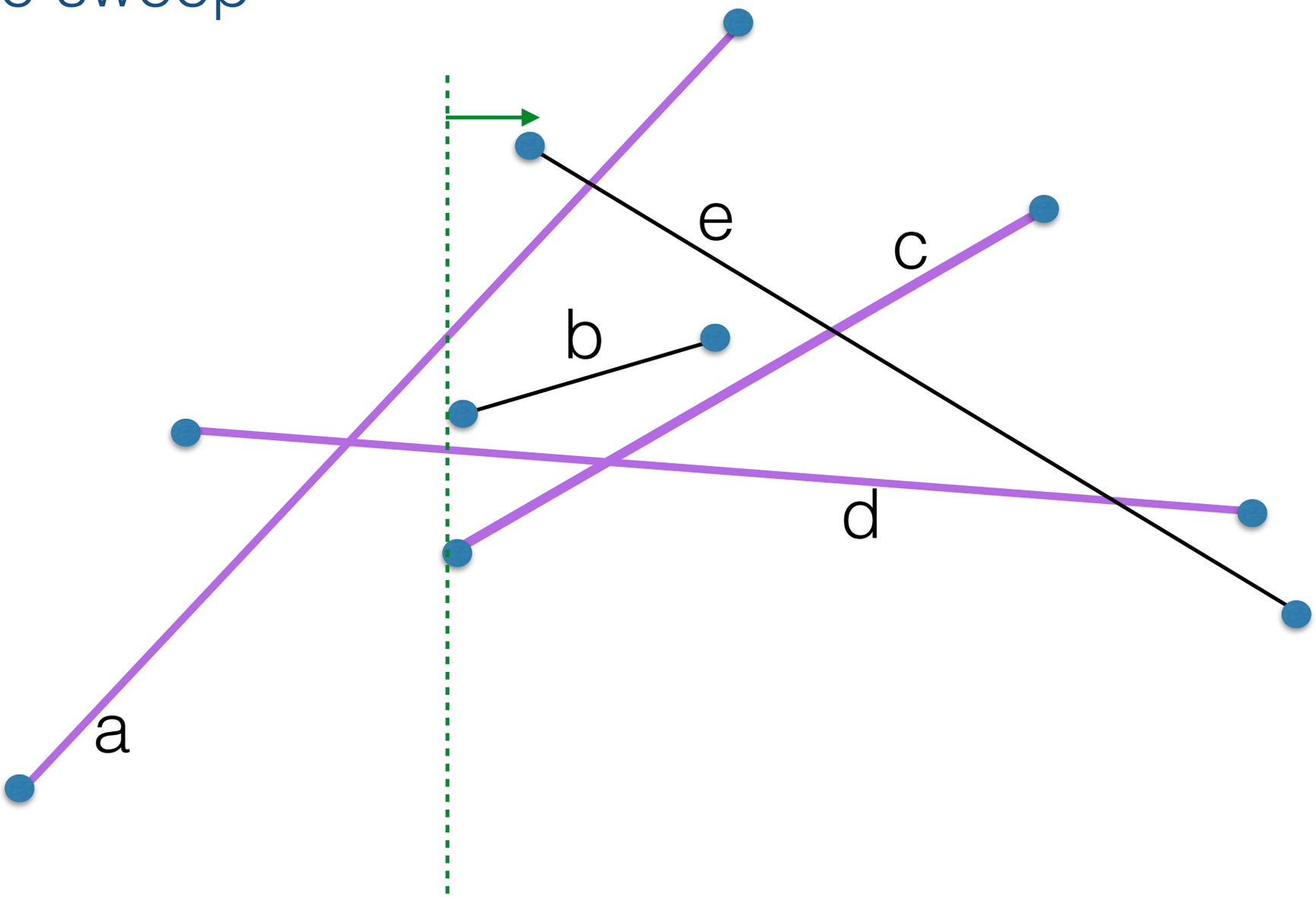
# The sweep




- Let X be the set of all x-coords of segments
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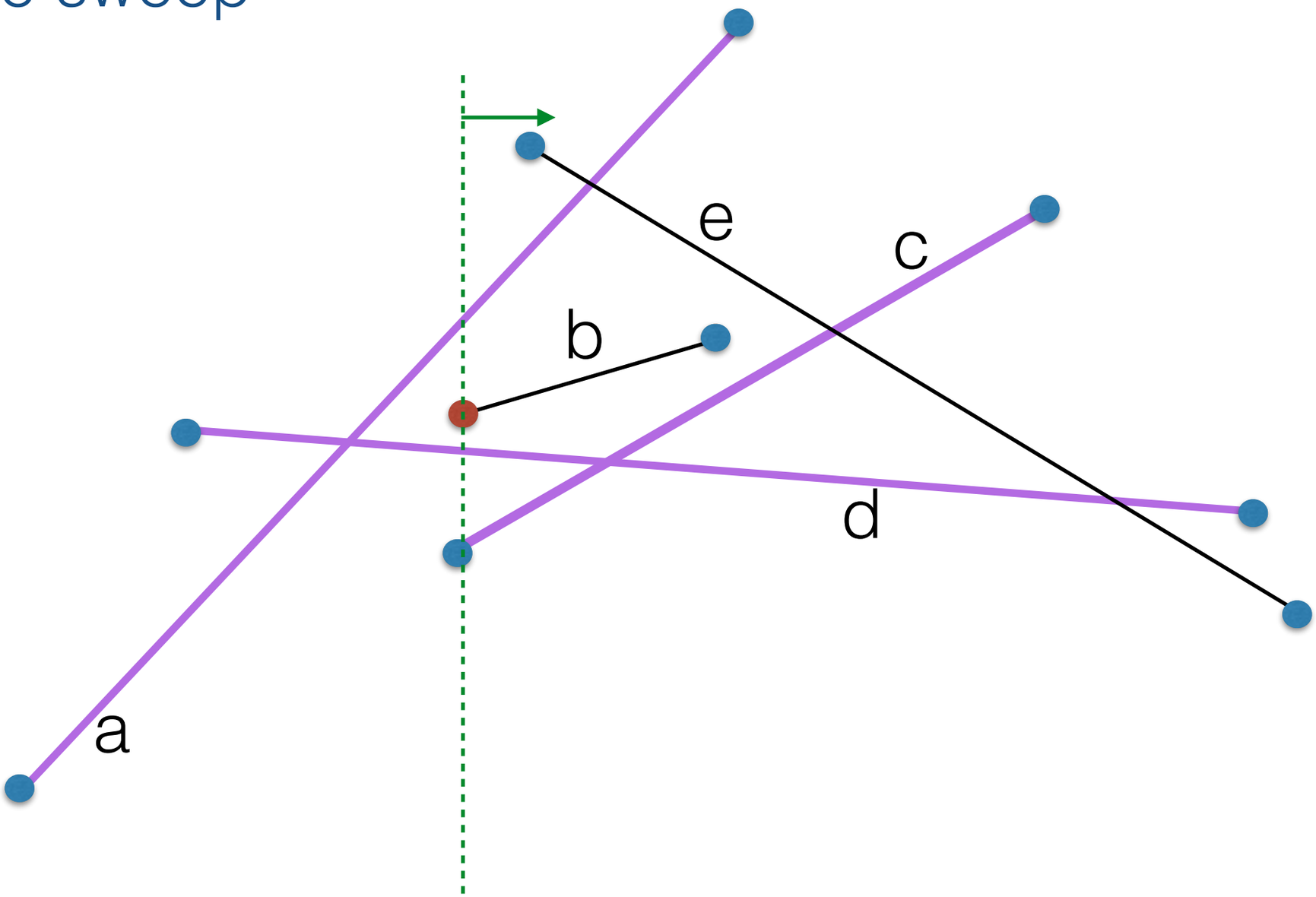
# The sweep

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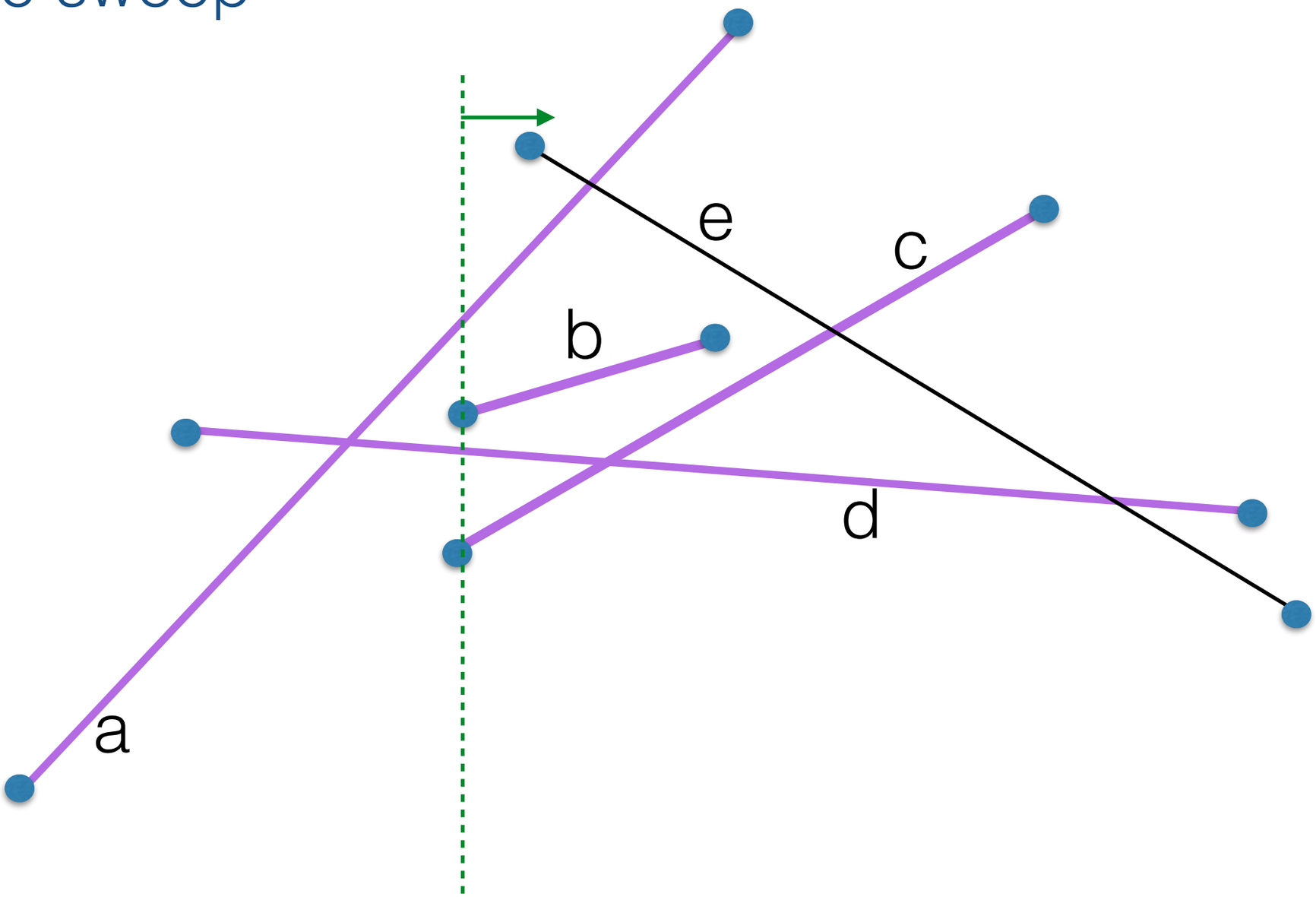


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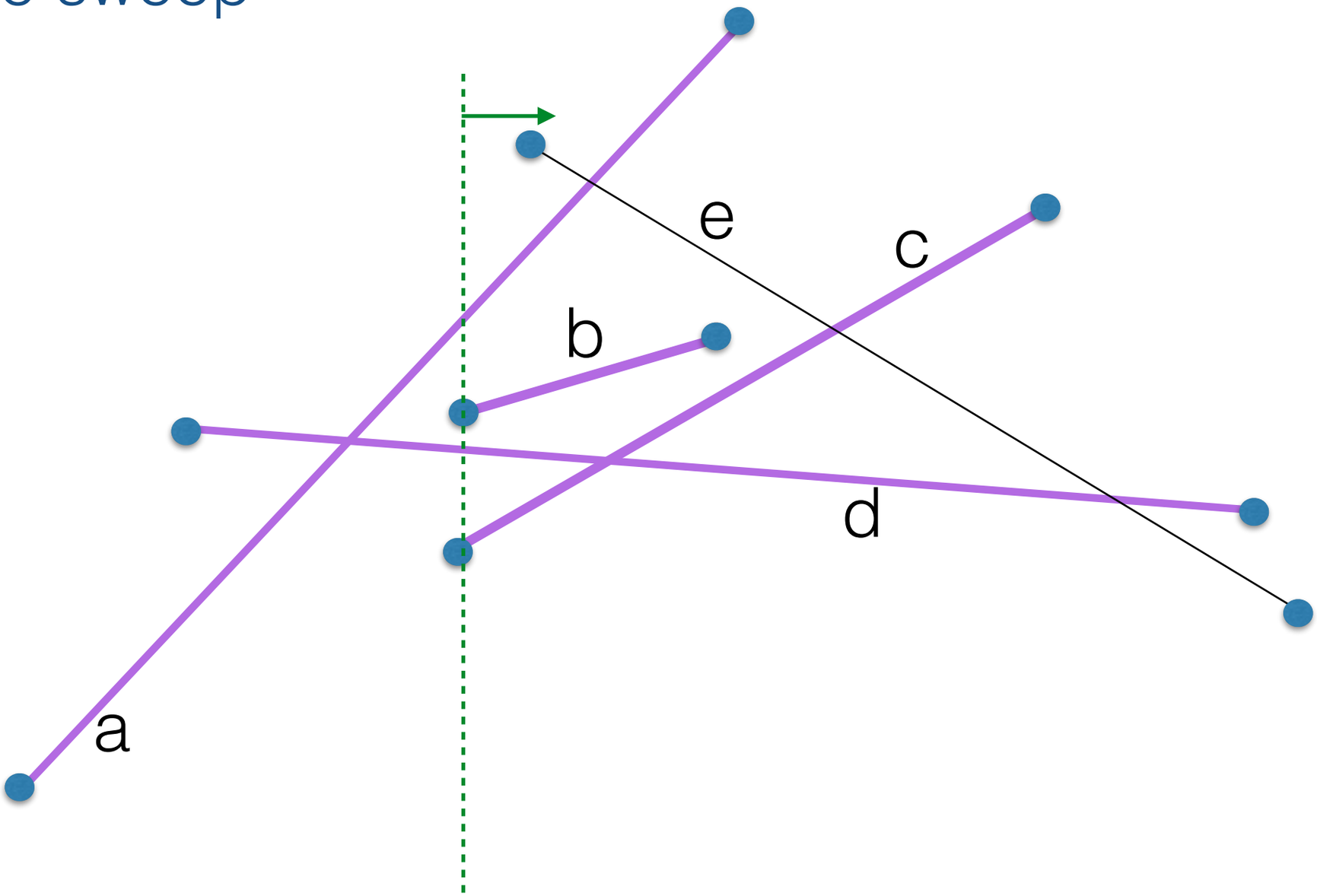
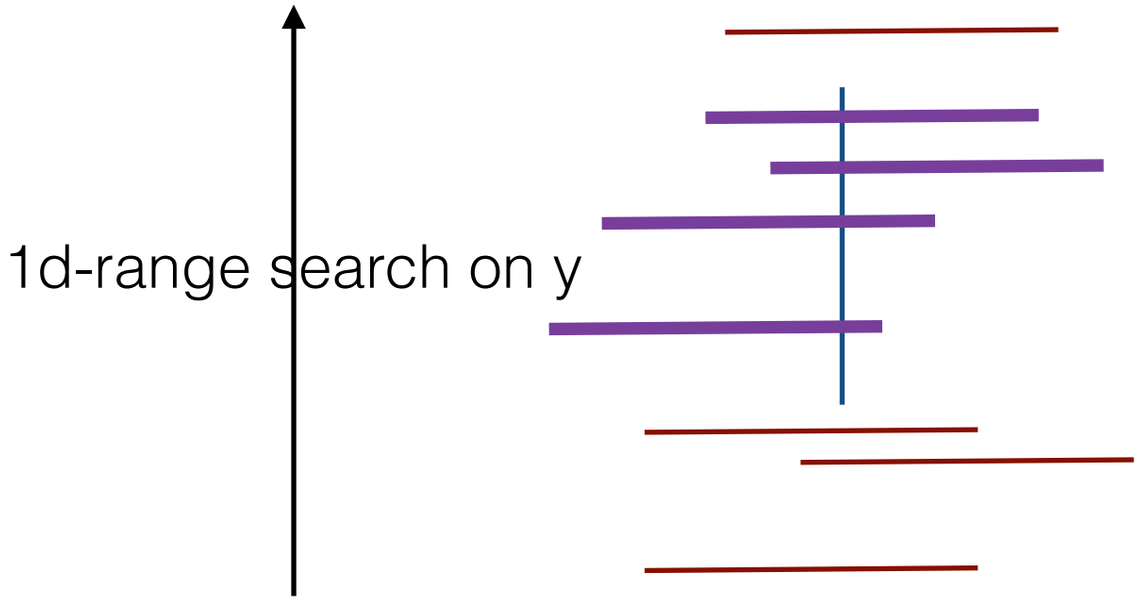
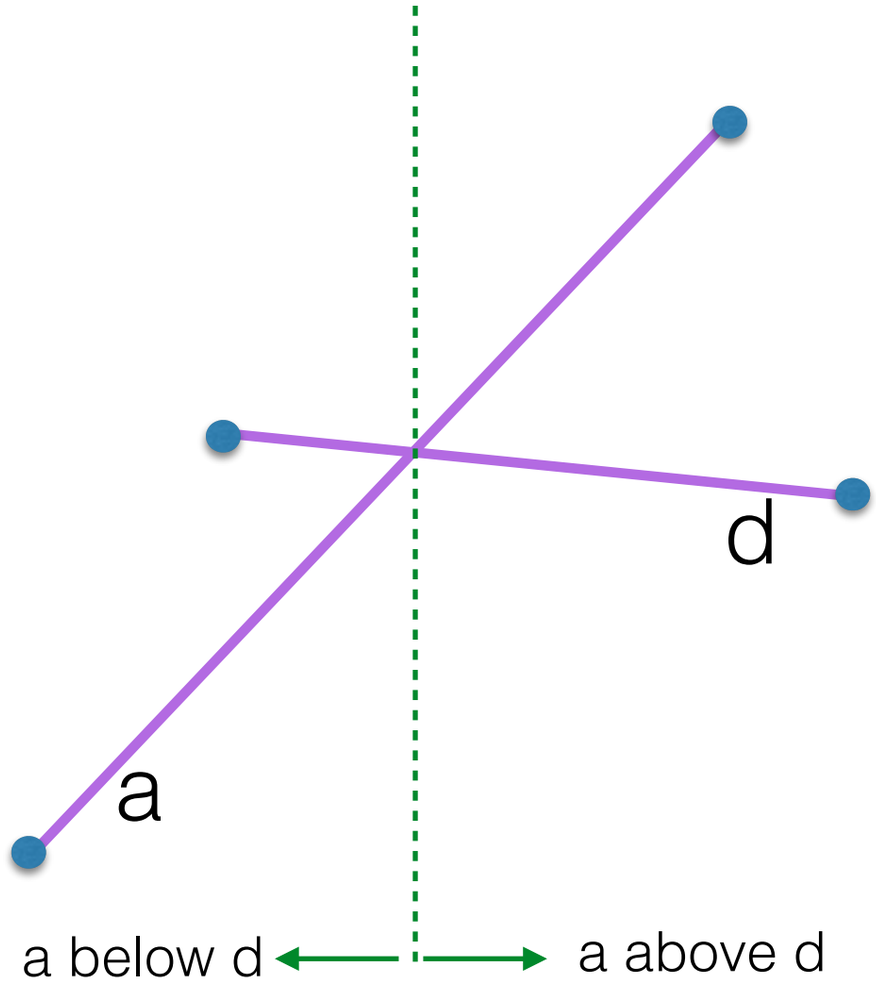



- 4 segments are active
- How do we order these segments?
- How do we detect intersections?

How do we order active segments?

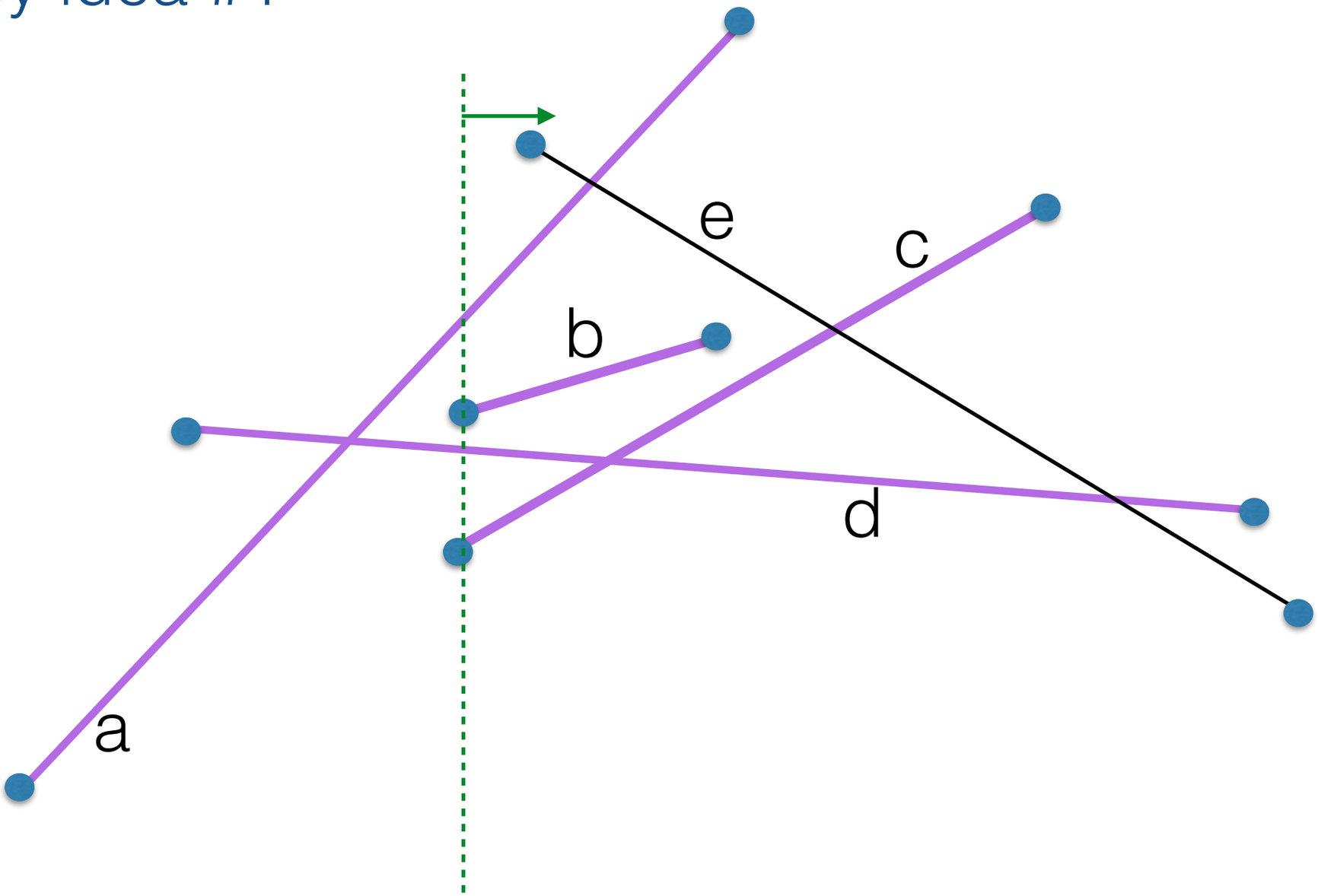



orthogonal segments



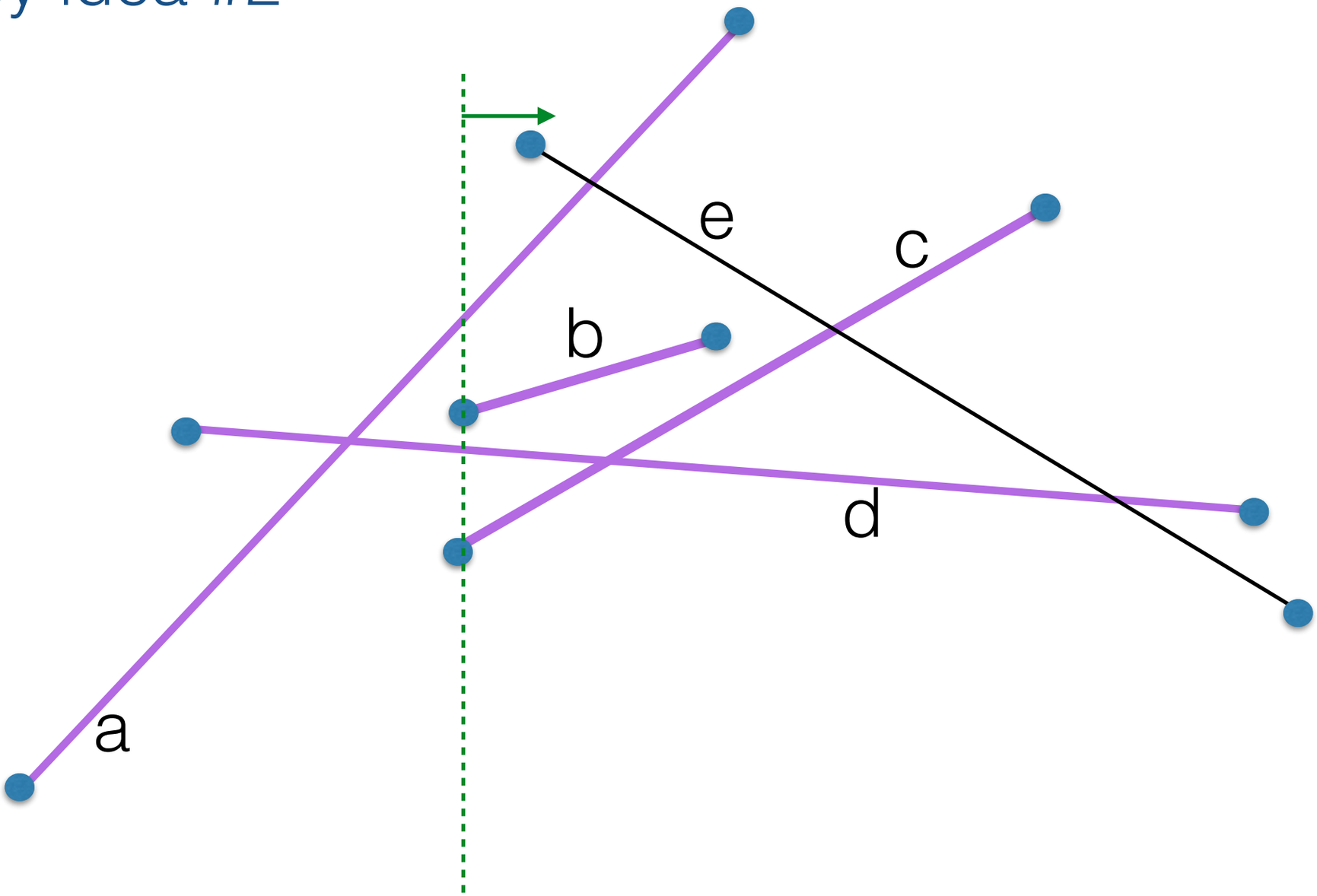

general segments

# Key idea #1



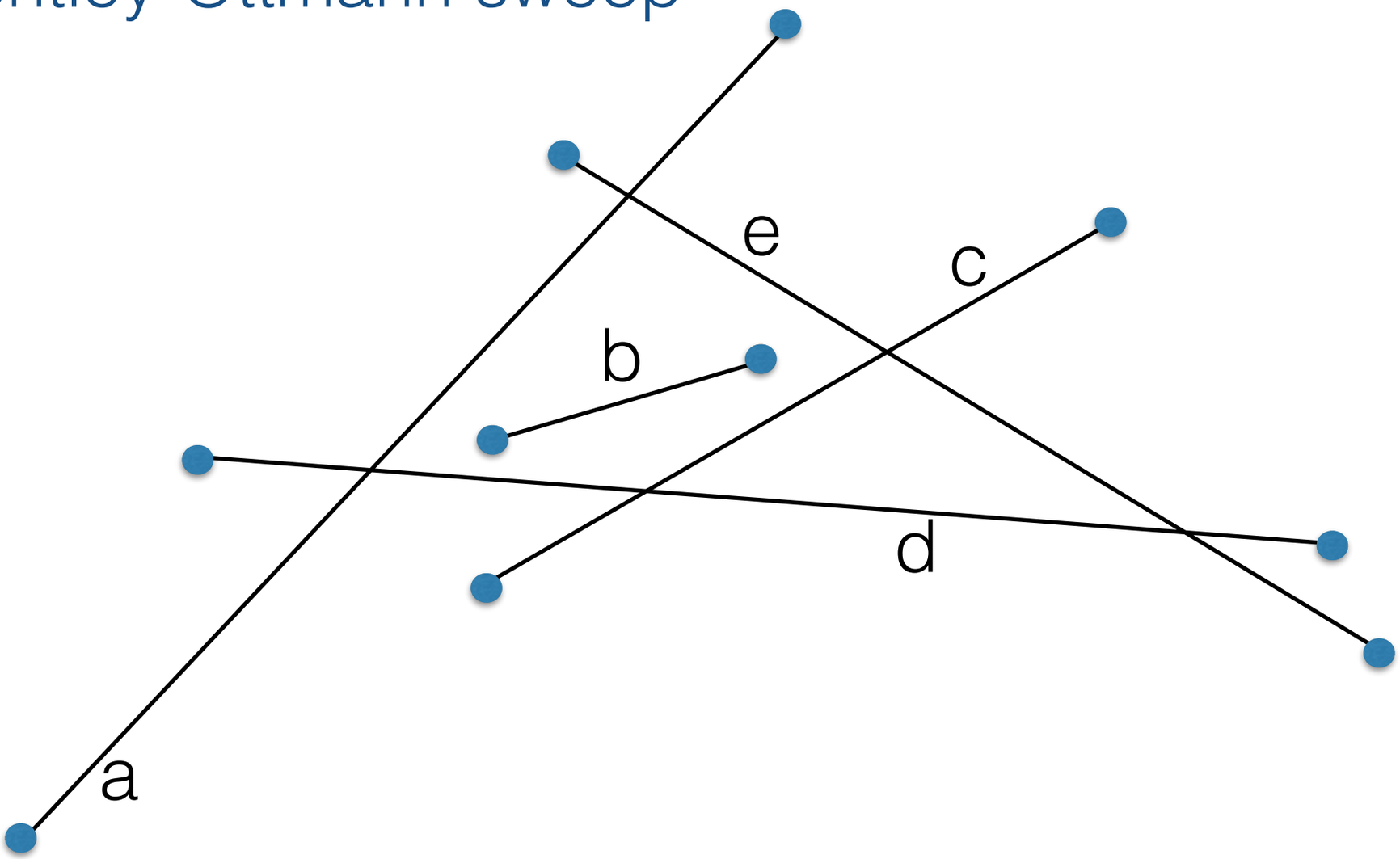

- Use above-below order
- Order will flip at intersection point

# Key idea #2



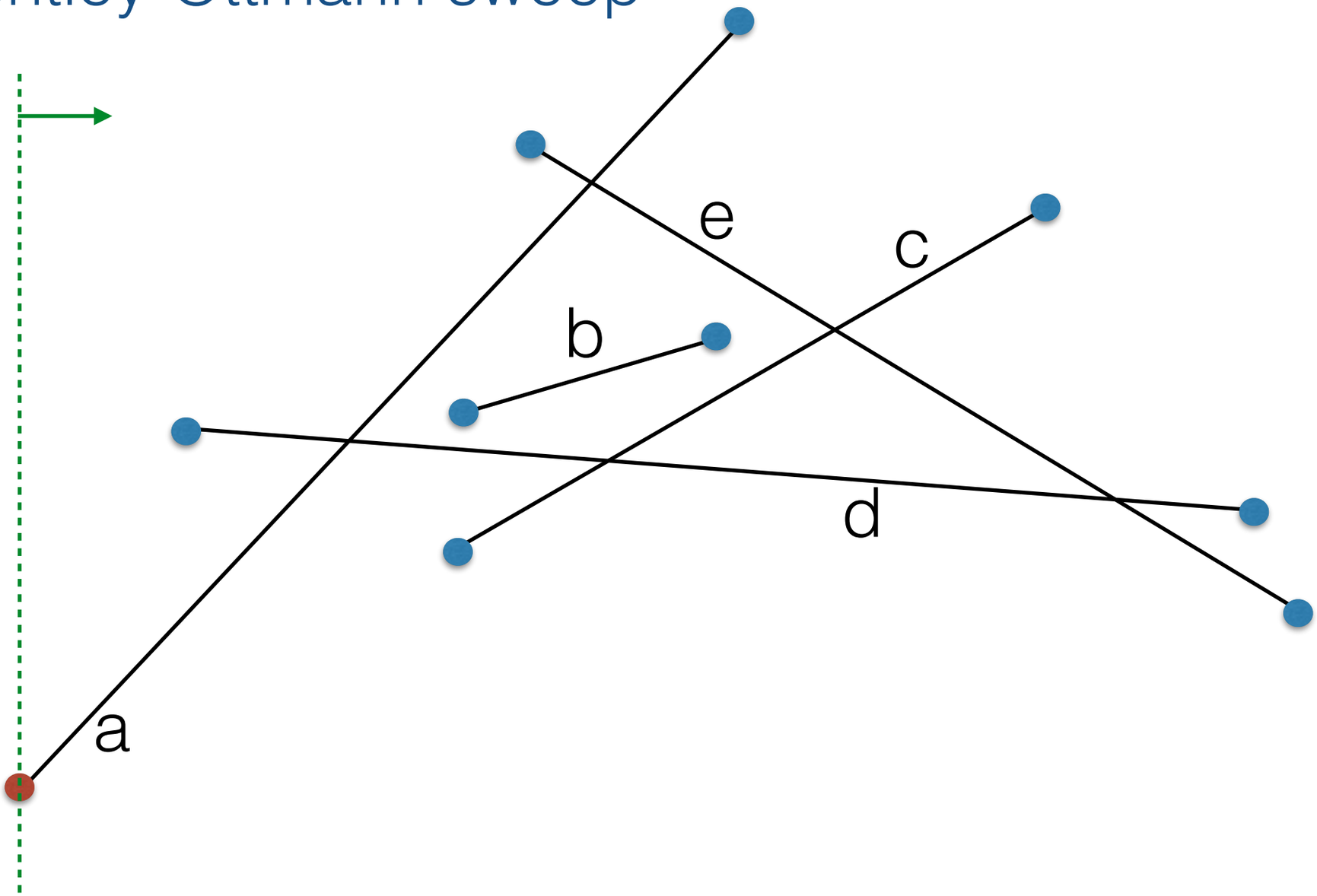

- Segments that intersect are consecutive in above-below order just before they intersect

# Bentley-Ottmann sweep




- Let X be the set of all x-coords of segments
- Initialize AS = {}
- Traverse events in order

# Bentley-Ottmann sweep



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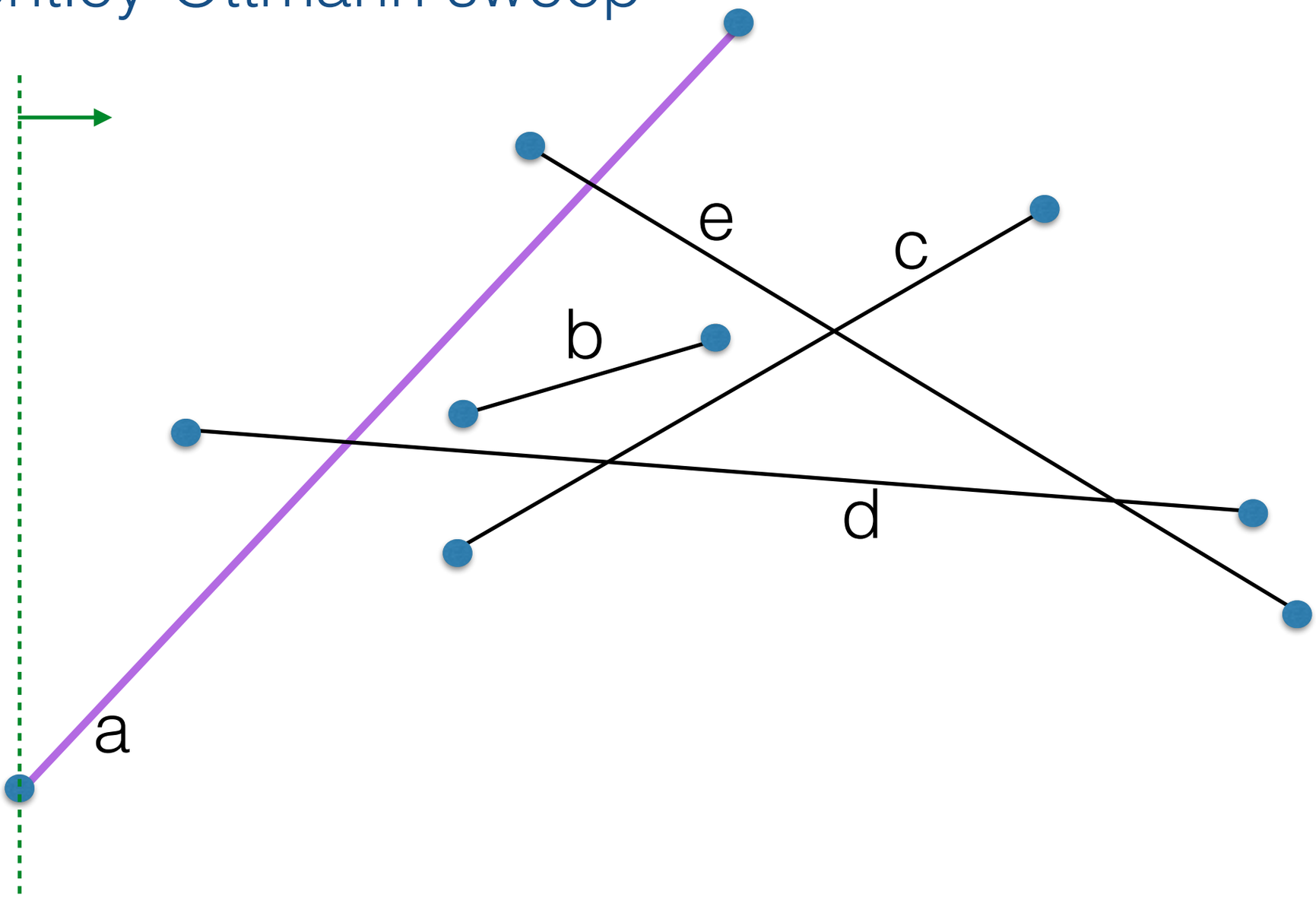
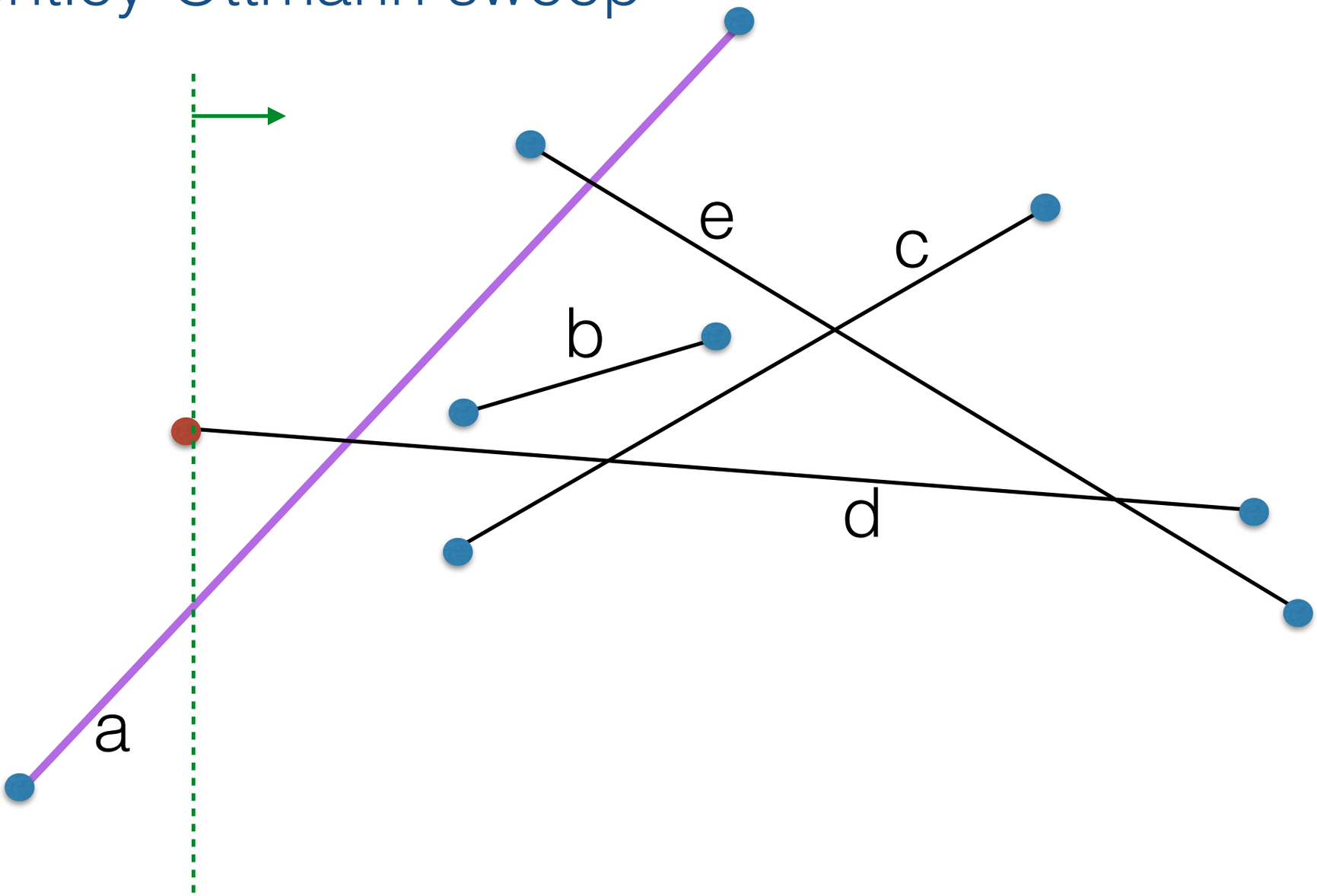



- this event is start of segment a:
  - insert a in AS: a

# Bentley-Ottmann sweep



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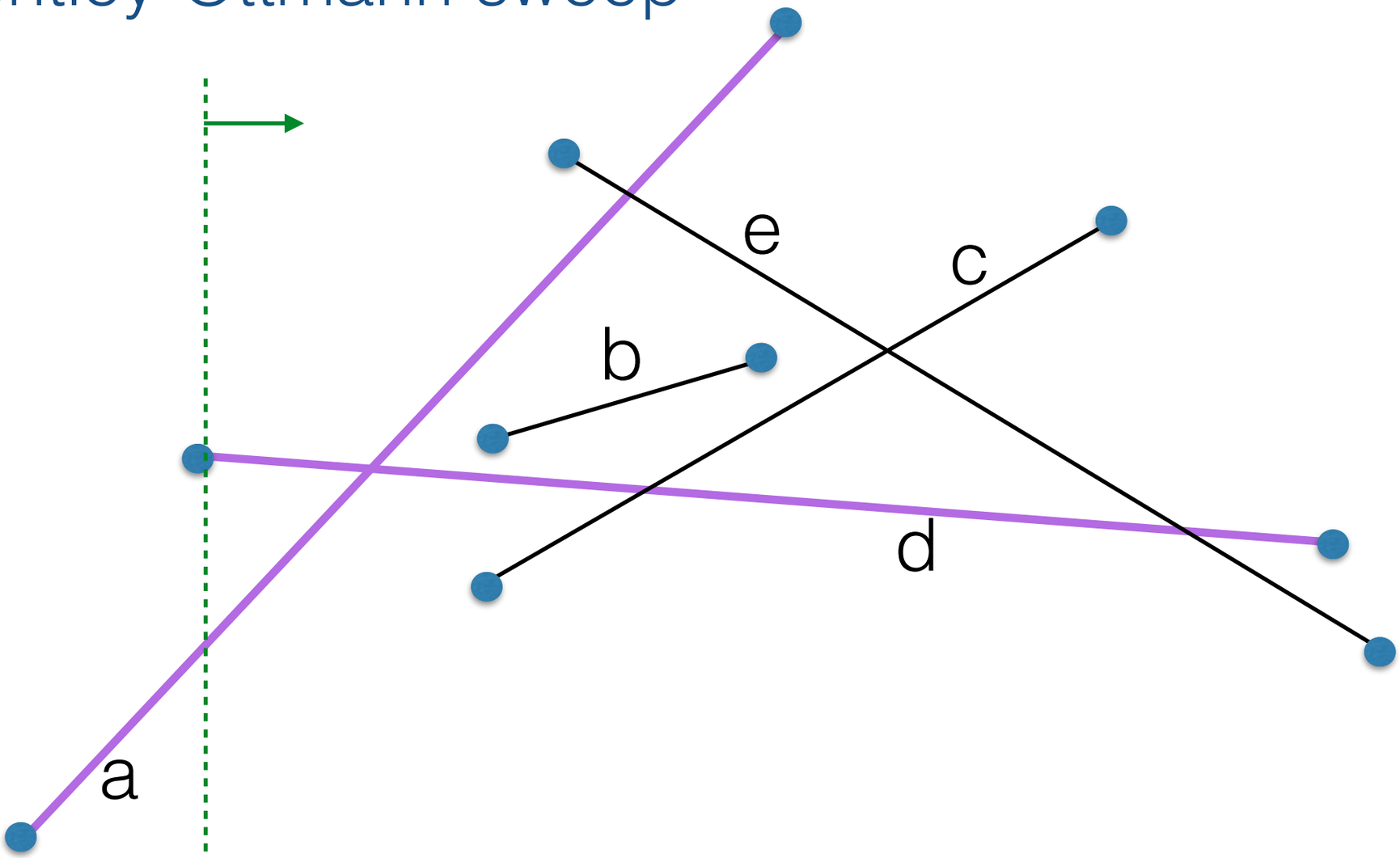
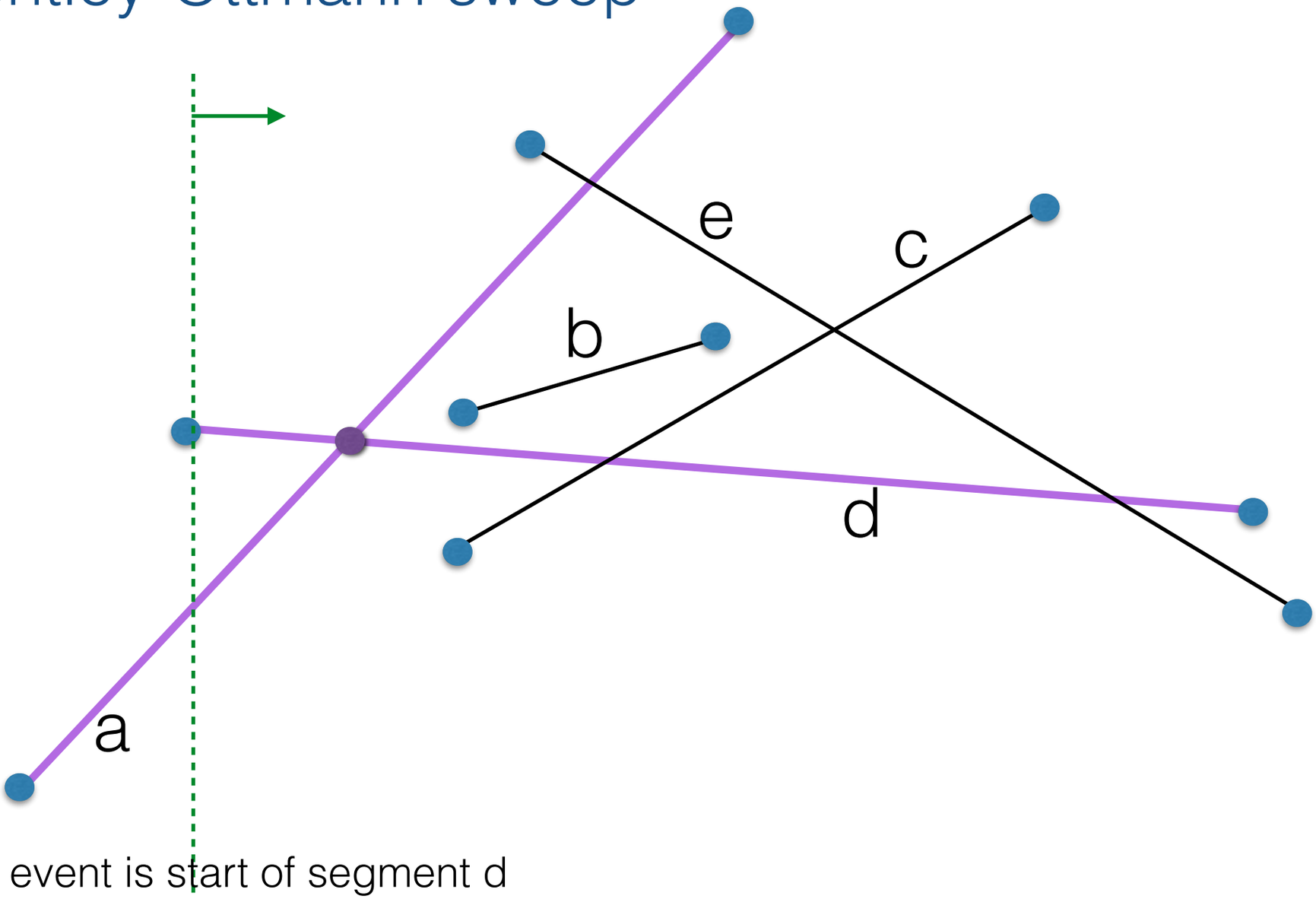


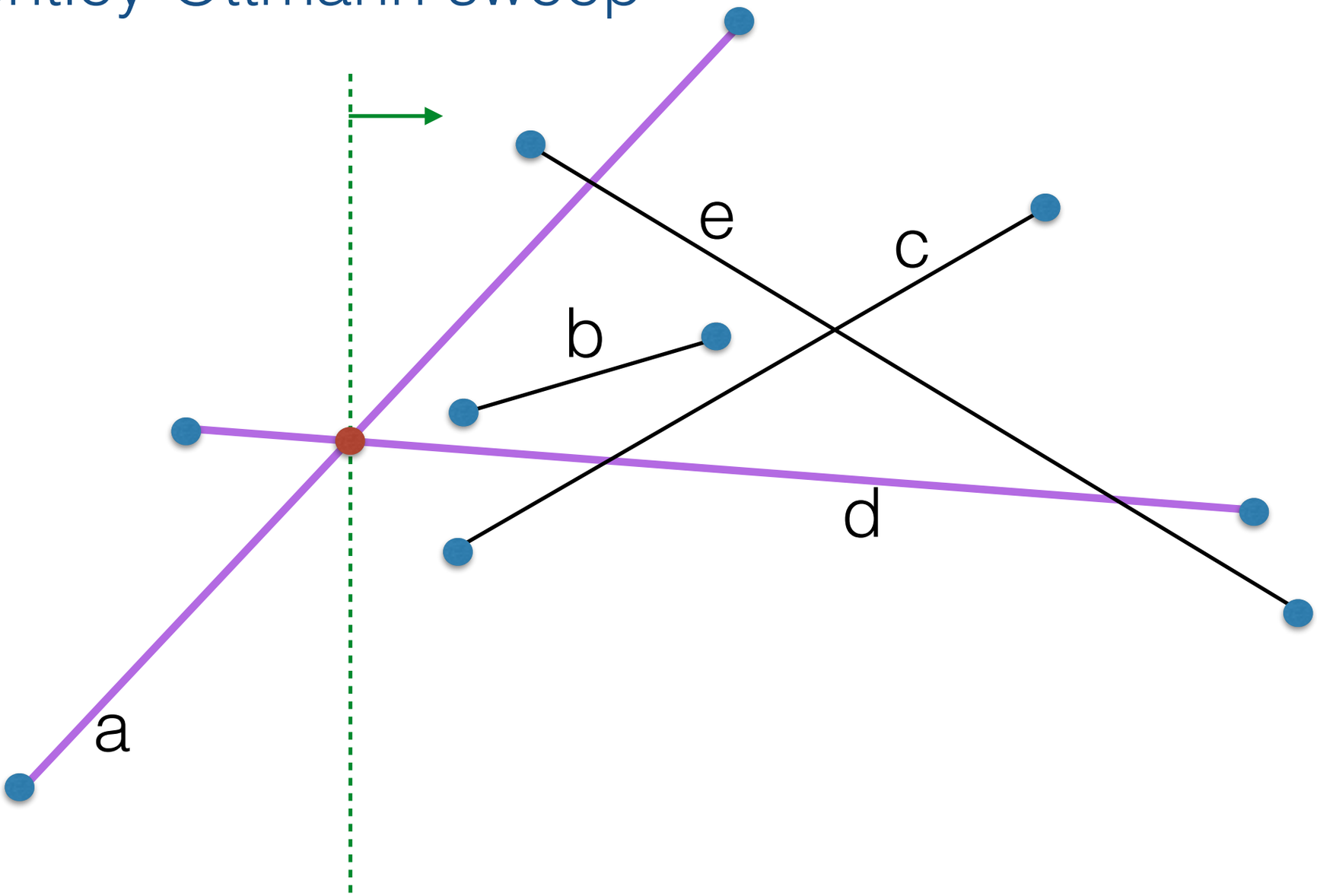

- this event is start of segment d
  - insert d in AS:  a < d
  - check if (d,a) intersect to the right of the line; they do; report  point  and insert it in the list of future events

# Bentley-Ottmann sweep




- this event is start of segment d
  - insert d in AS, d is above a (a < d)
  - check if (d,a) intersect to the right of the line; they do; report point and insert it in the list of future events

# Bentley-Ottmann sweep



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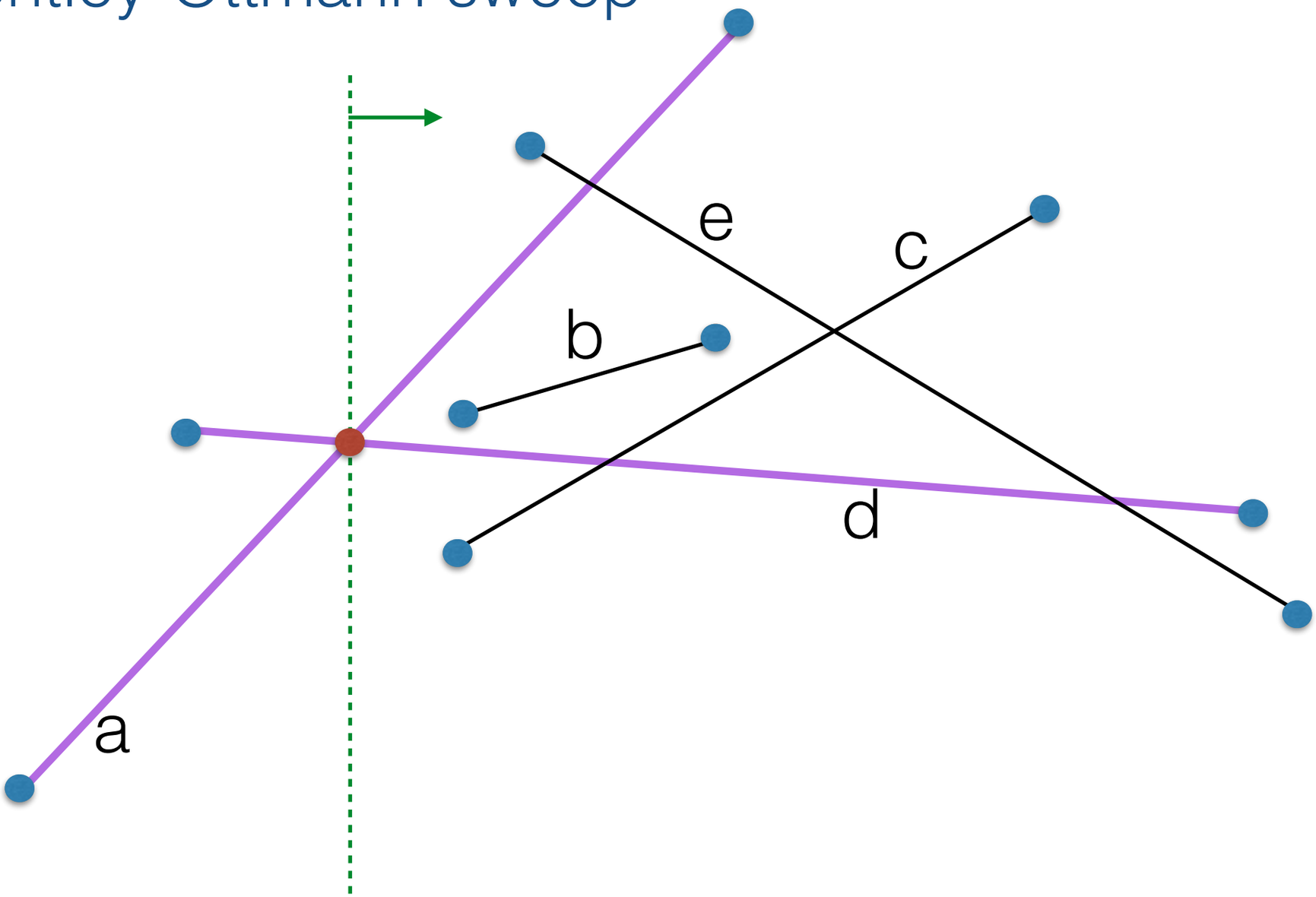
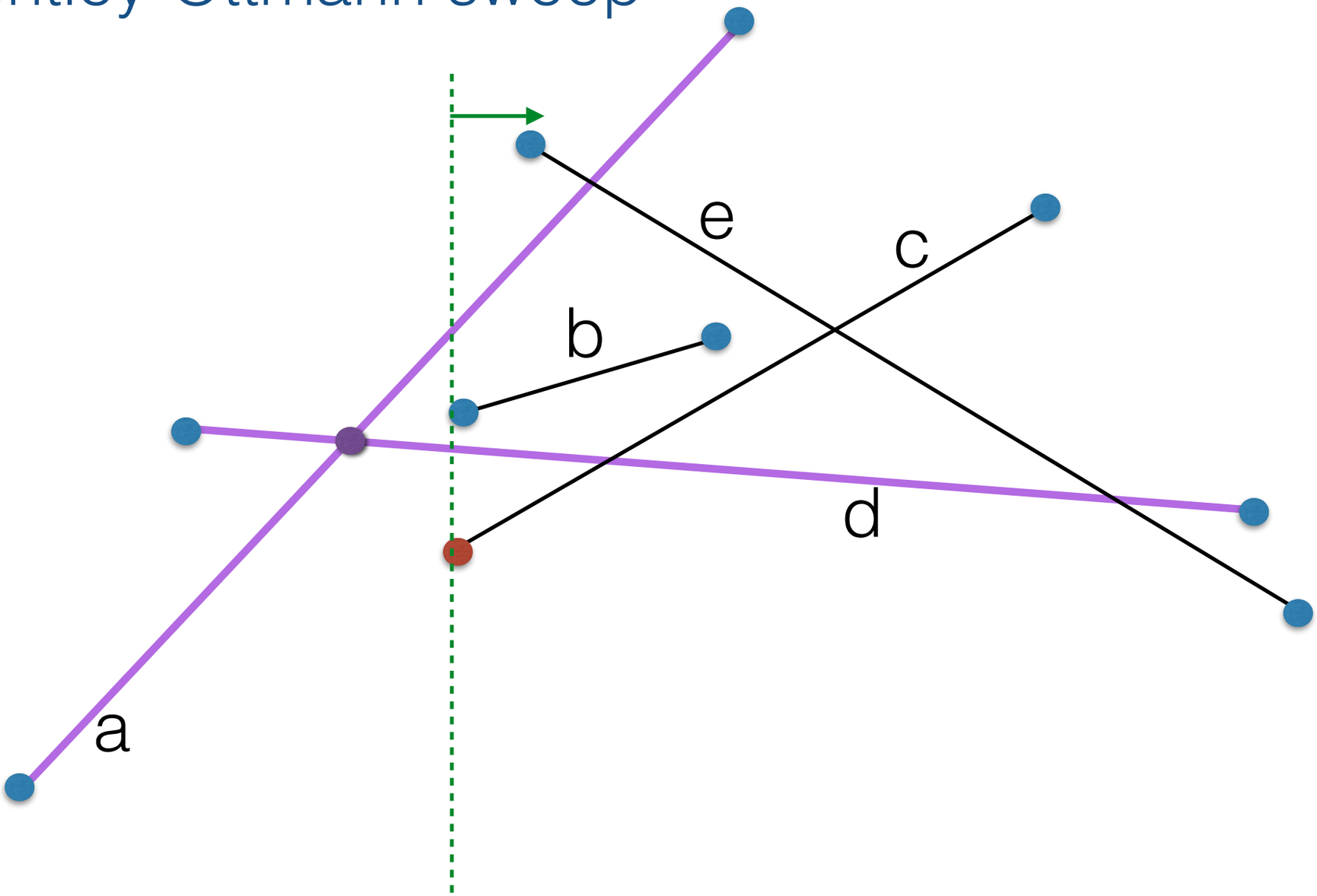



- this event is an intersection point of (a,d):
  - flip a and d is AS: a is now above d (d < a)

# Bentley-Ottmann sweep



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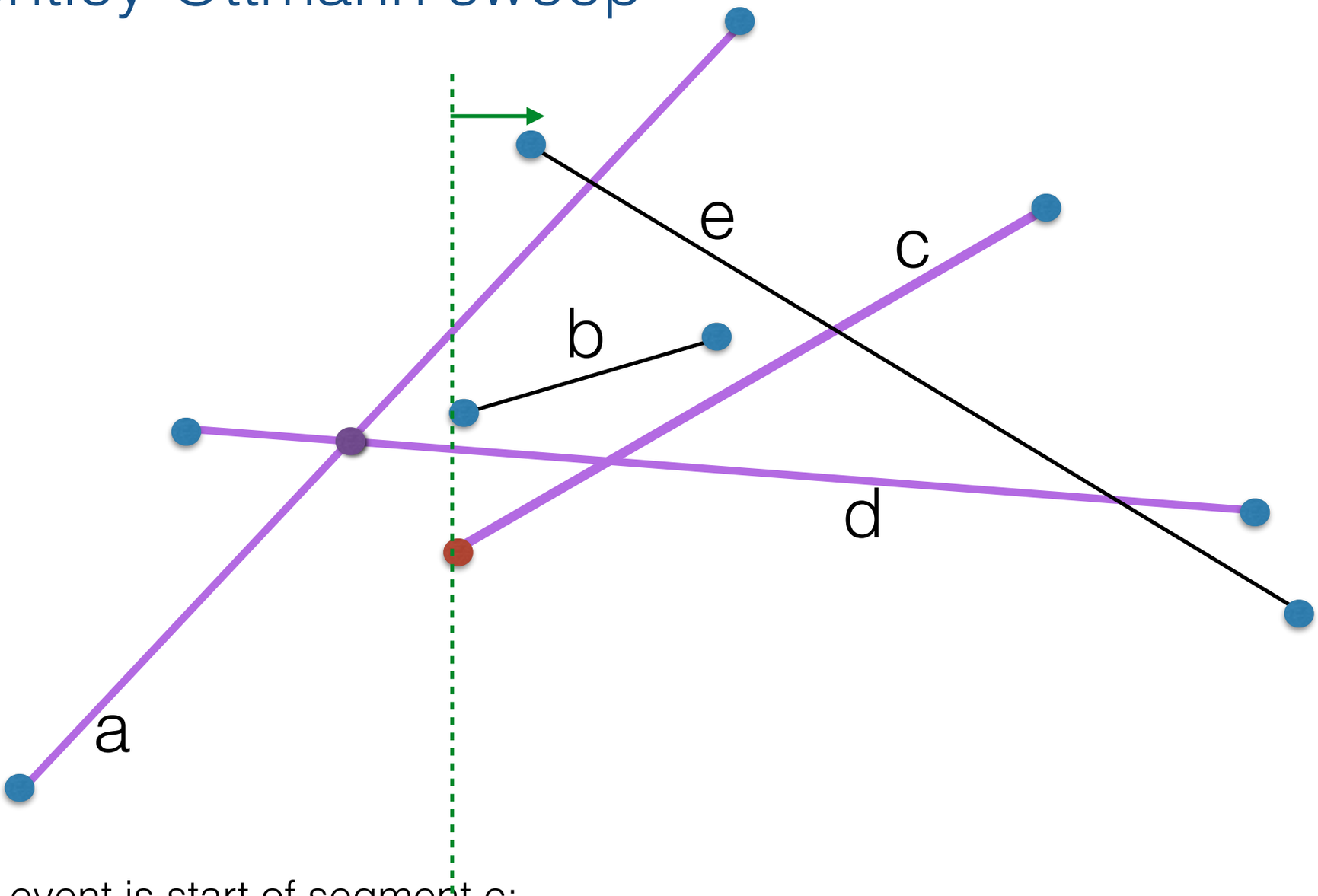
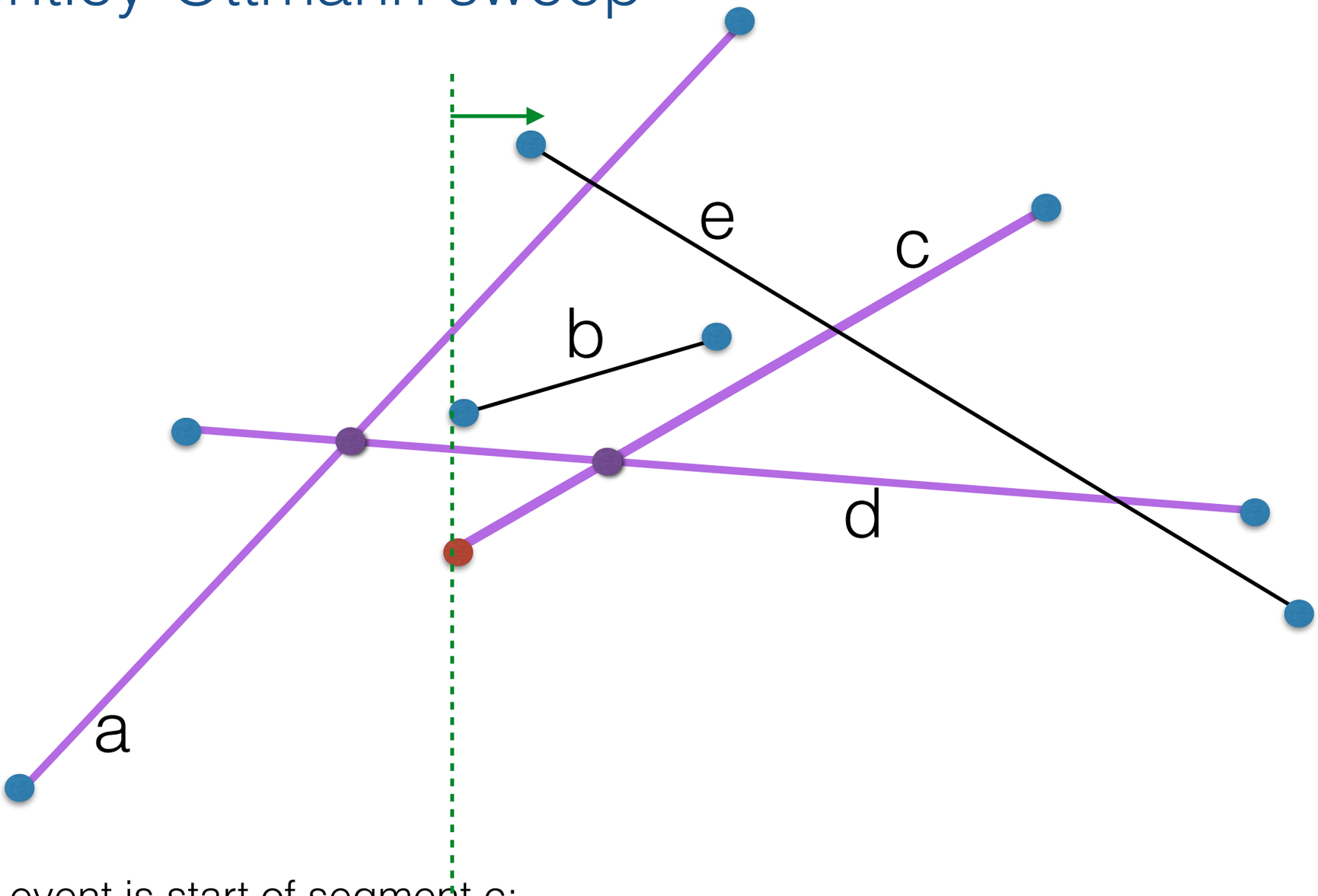



- this event is start of segment c:
  - insert c in AS; c is below d (c < d < a)
  - check c with its above and below neighbors for intersection to the right of the sweep line; this detects the intersection point of c and d; report it and insert it as future event

# Bentley-Ottmann sweep

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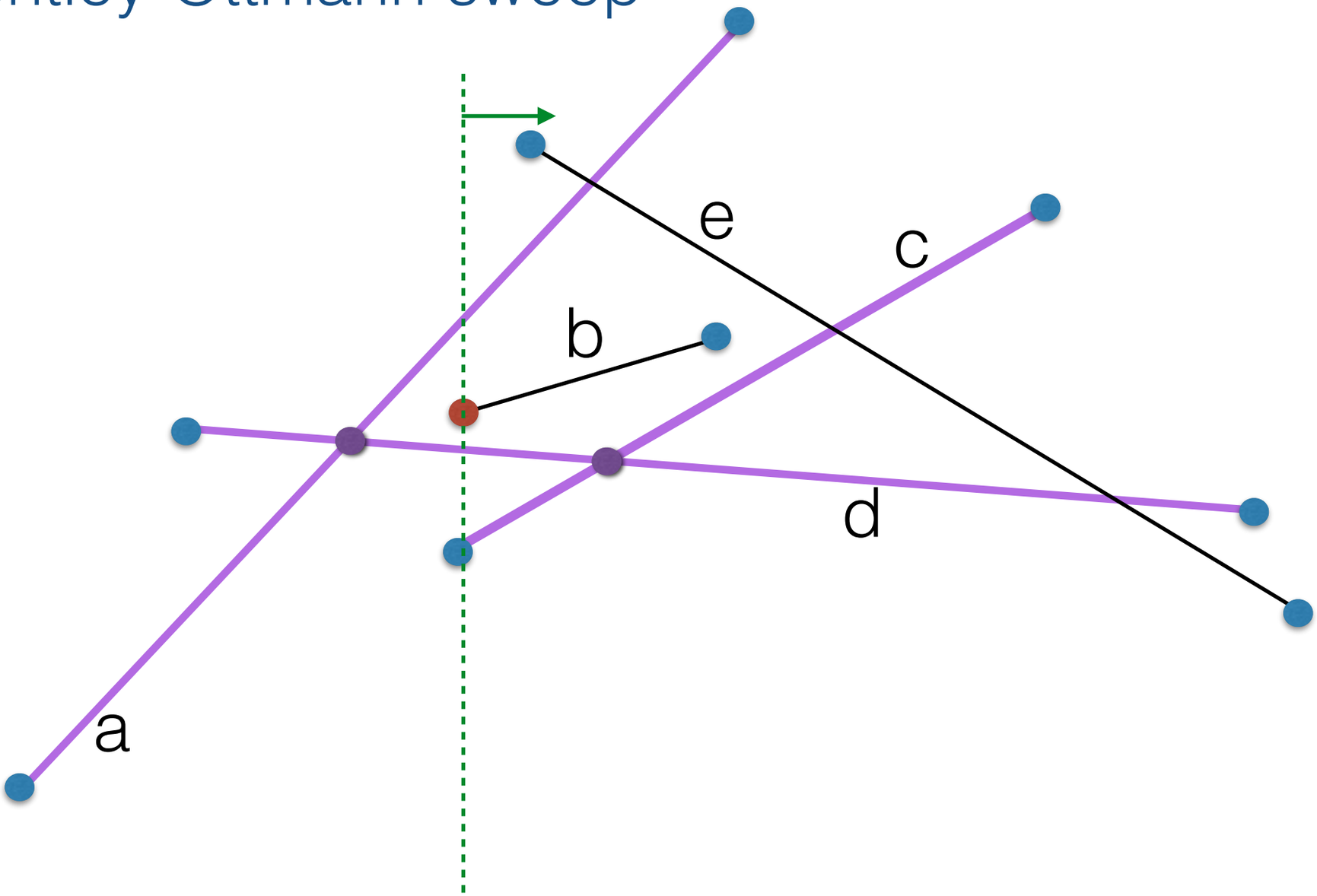
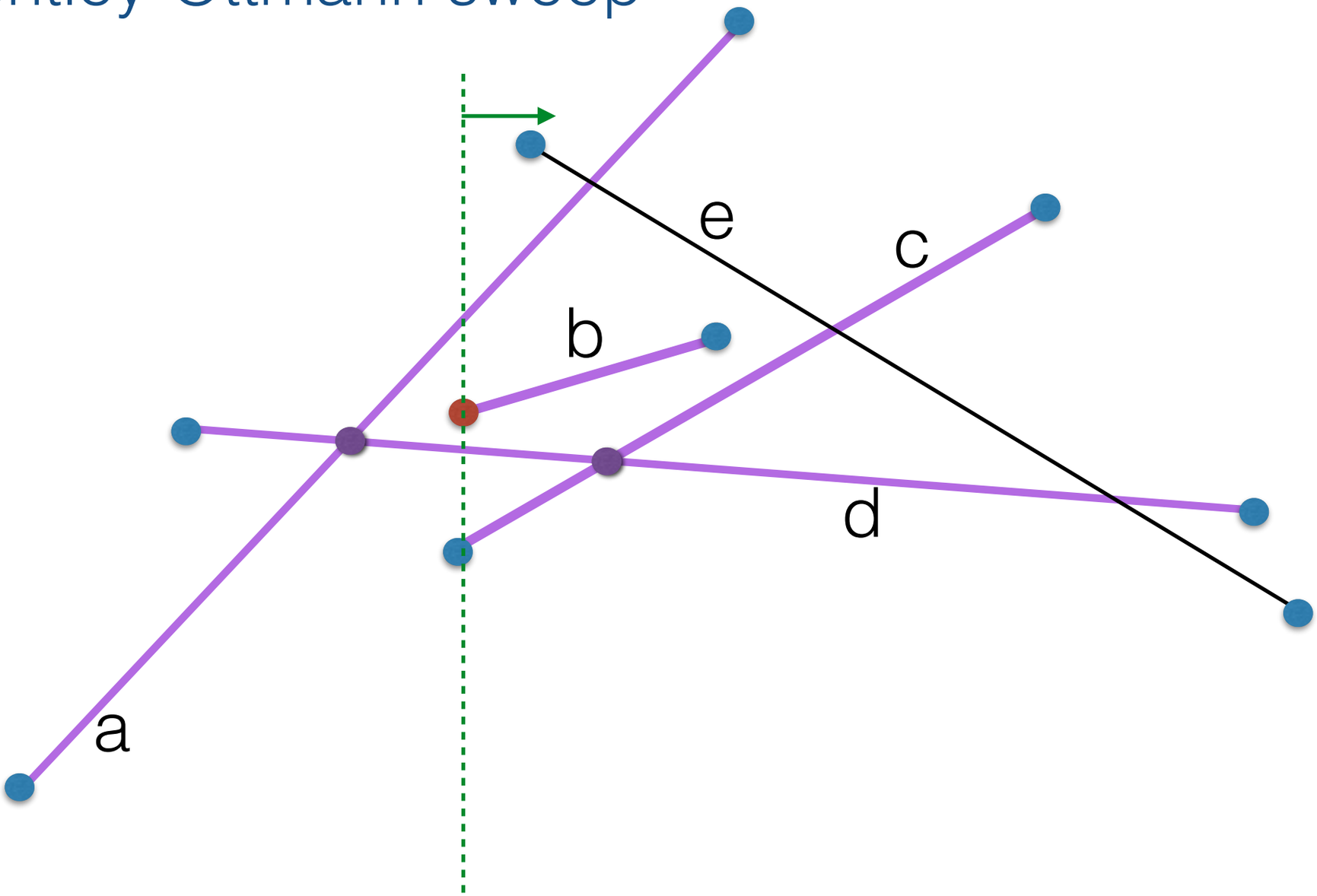


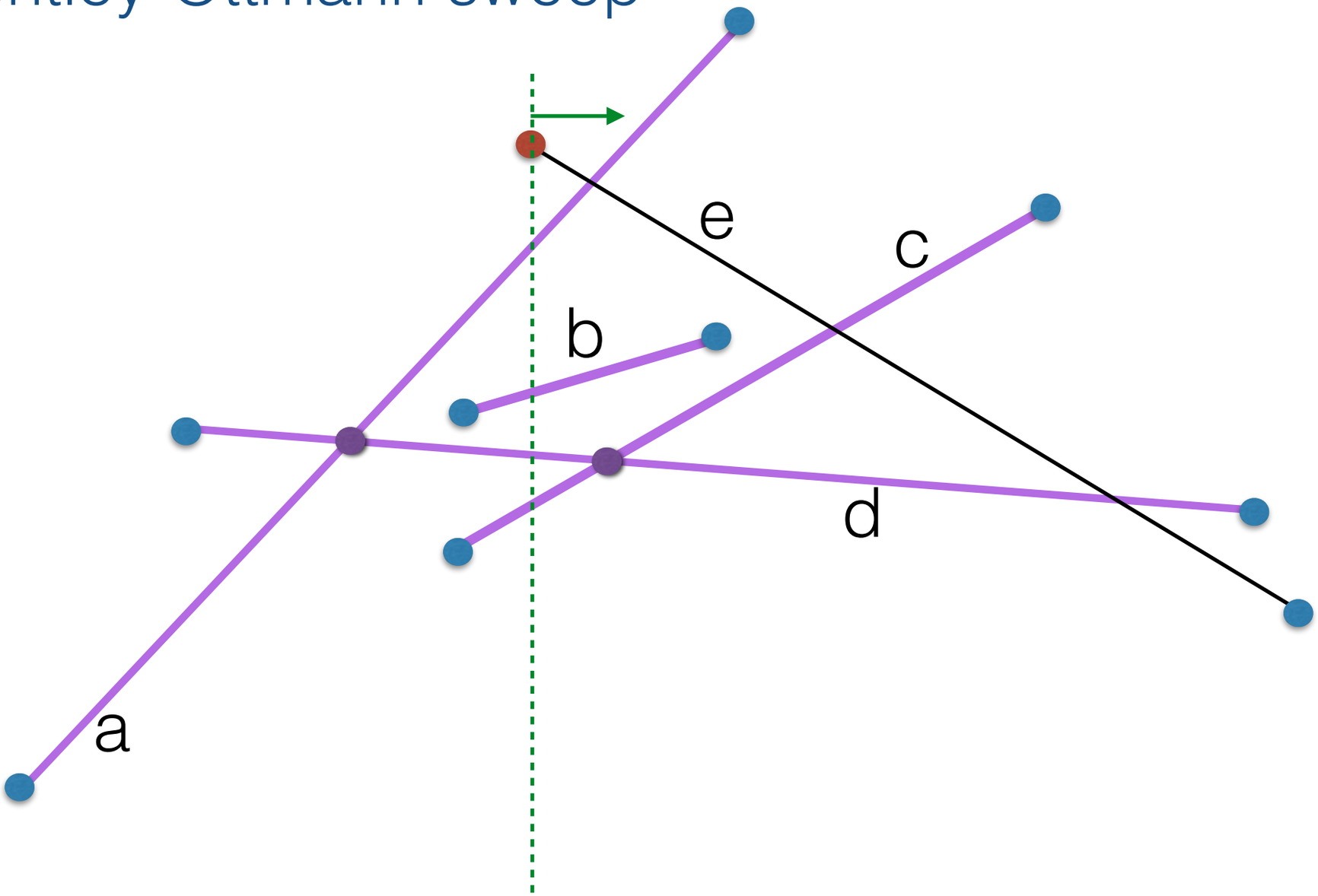

- this event is start of segment b:

# Bentley-Ottmann sweep



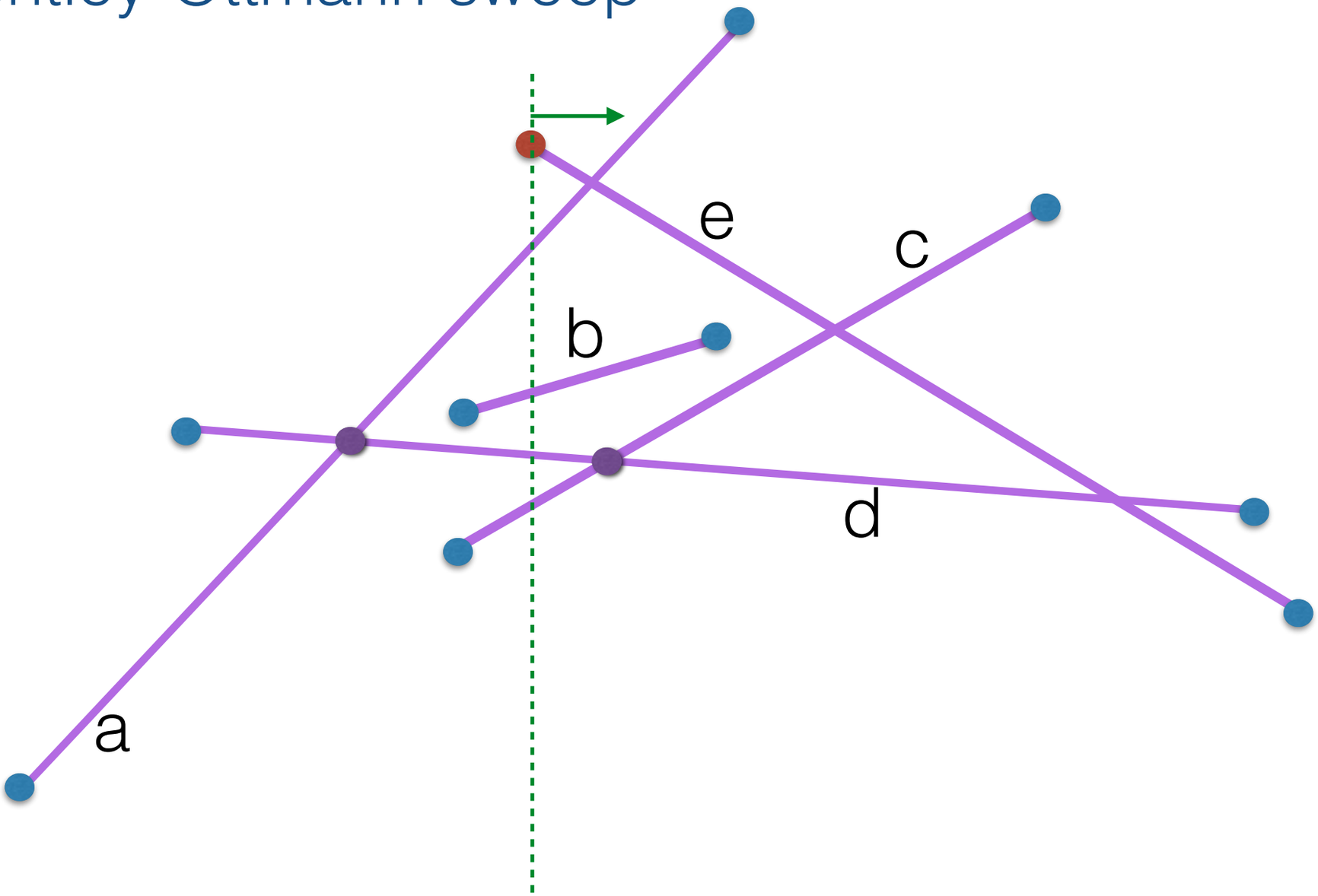

- this event is start of segment b:
  - insert b in AS; c < d < b < a
  - check b with its above and below neighbors for intersection to the right of the sweep line; (d,b) don't intersect; (b, a) don't intersect

# Bentley-Ottmann sweep



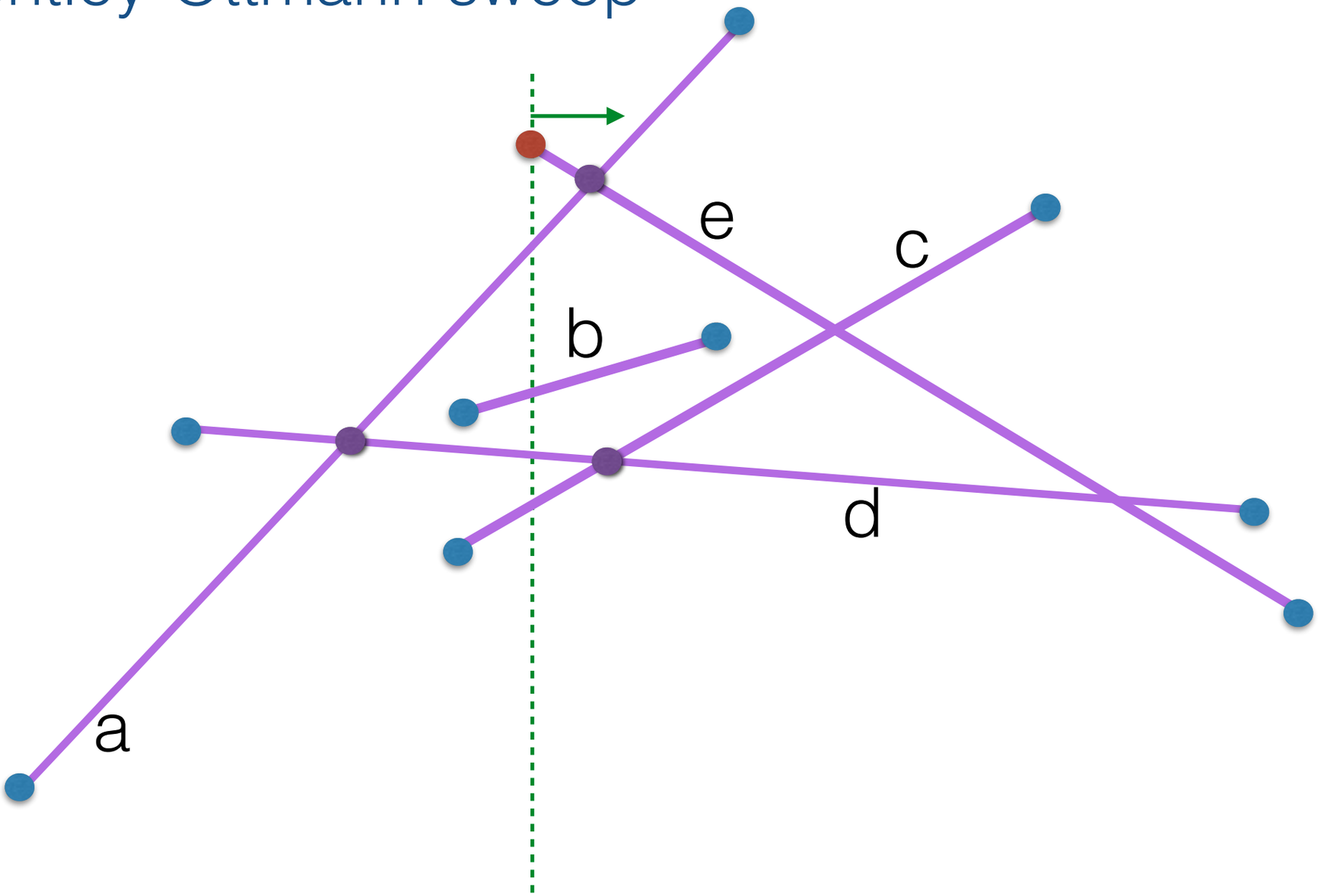

- this event is start of segment e:

# Bentley-Ottmann sweep




- this event is start of segment e:
  - insert e in AS: c < d < b < a < e
  - check e with its above and below neighbors for intersection to the right of the sweep line; this detects intersection point of (a,e); report it and insert it as future event

# Bentley-Ottmann sweep

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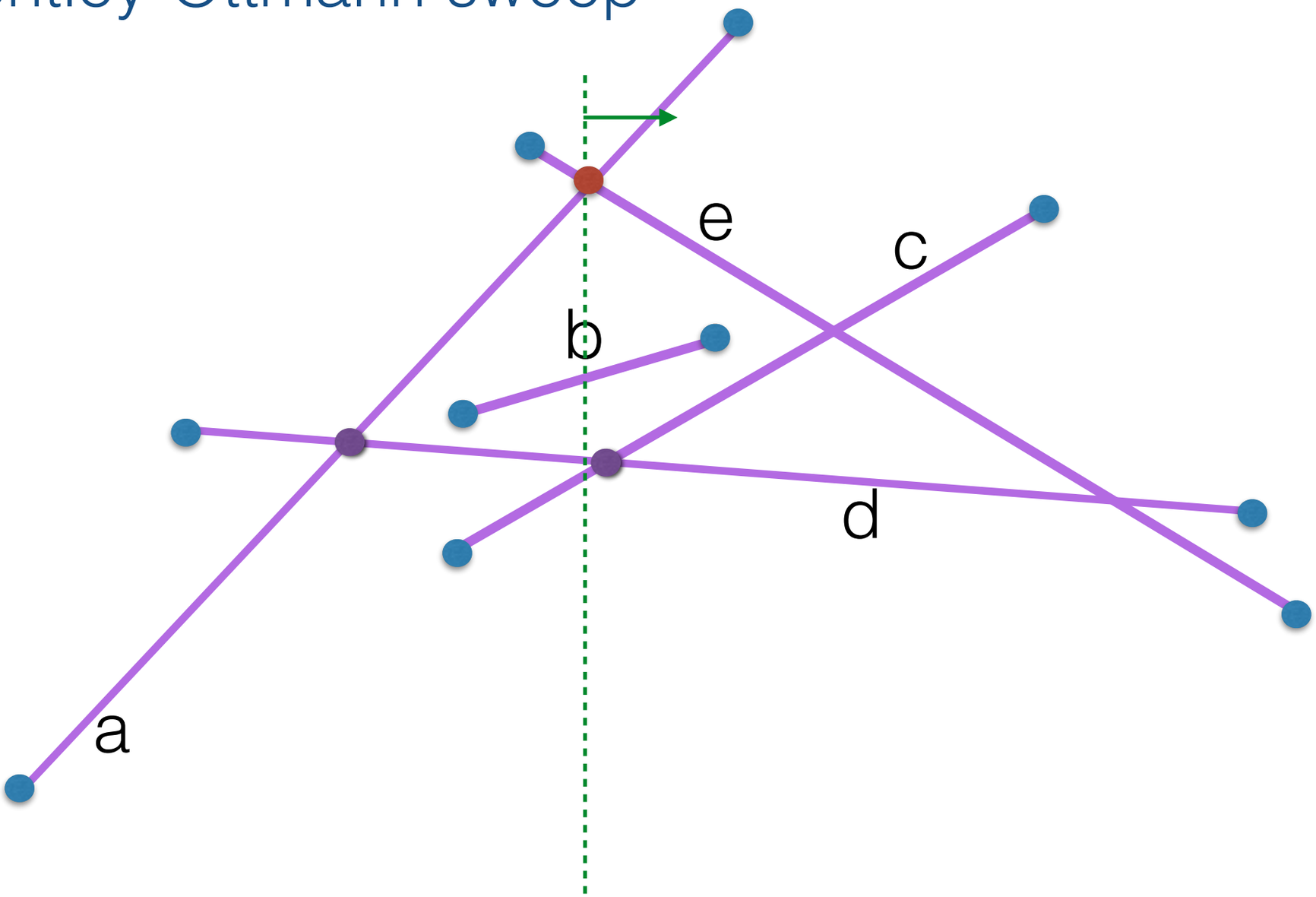
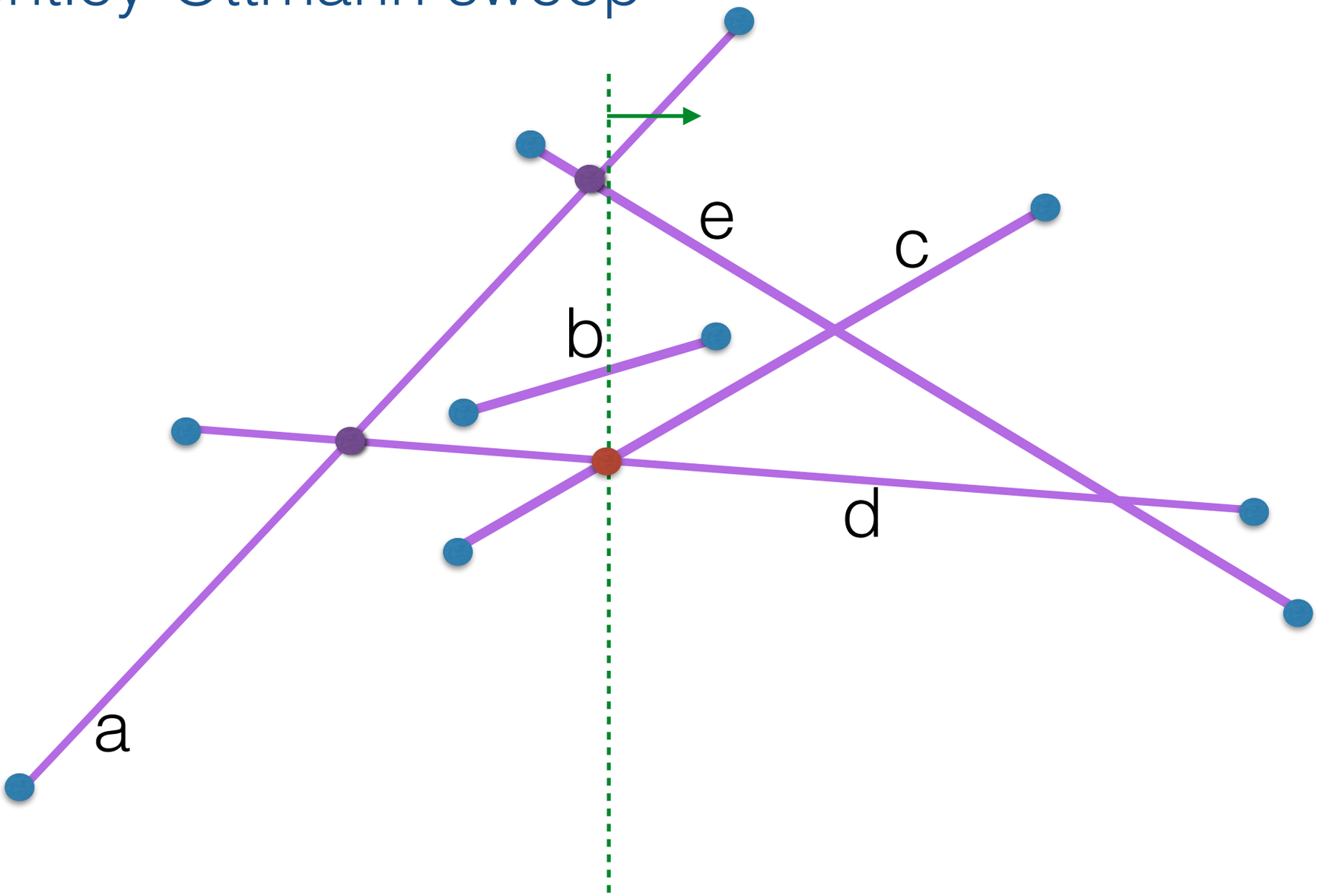


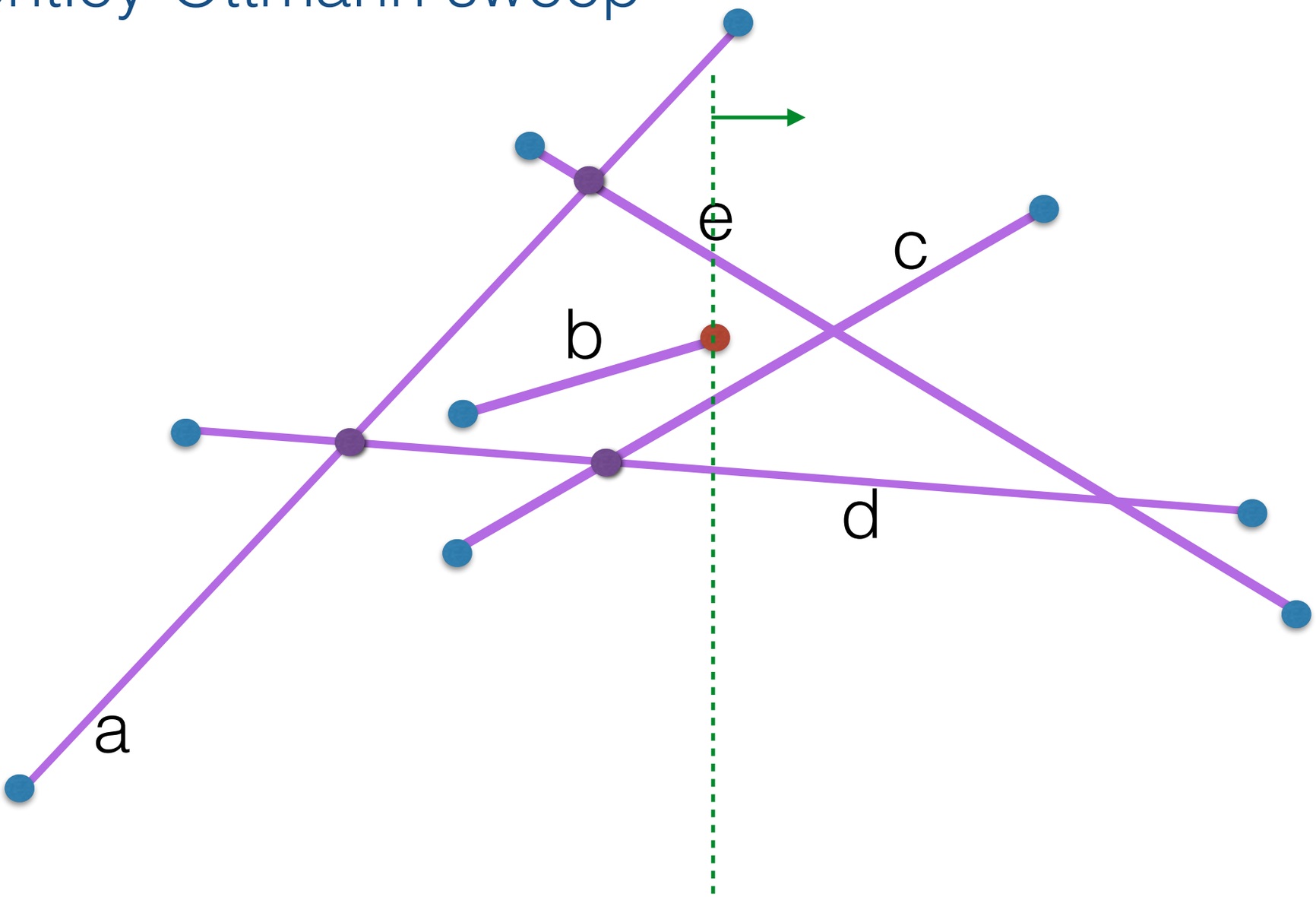

- this event is intersection of (a,e):
  - flip a and e: c < d < b < e < a
  - check new neighbors (e,b) for intersection to the right of the sweep line; (e,b) don’t intersect

# Bentley-Ottmann sweep



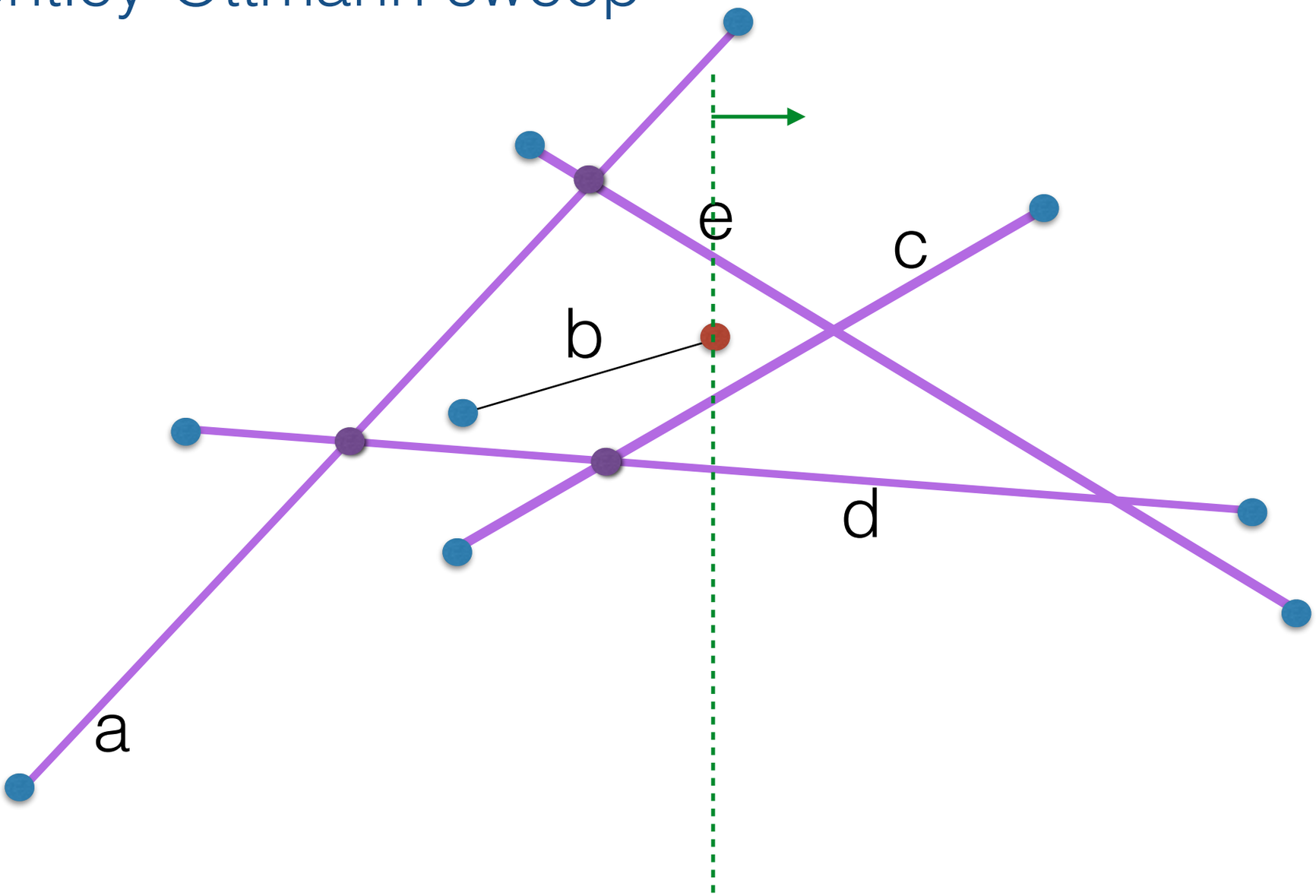

- this event is intersection of (c,d):
  - flip c and d: d < c < b < e < a
  - check new neighbors (c,b) for intersection to the right of the sweep line; (c,b) don’t intersect

# Bentley-Ottmann sweep




- this event is end of segment b:
  - delete b from AS: d < c < e < a
  - check new neighbors (c,e) for intersection to the right of the sweep line; this detects the intersection point of (c,e); report it and insert it as future event

# Bentley-Ottmann sweep

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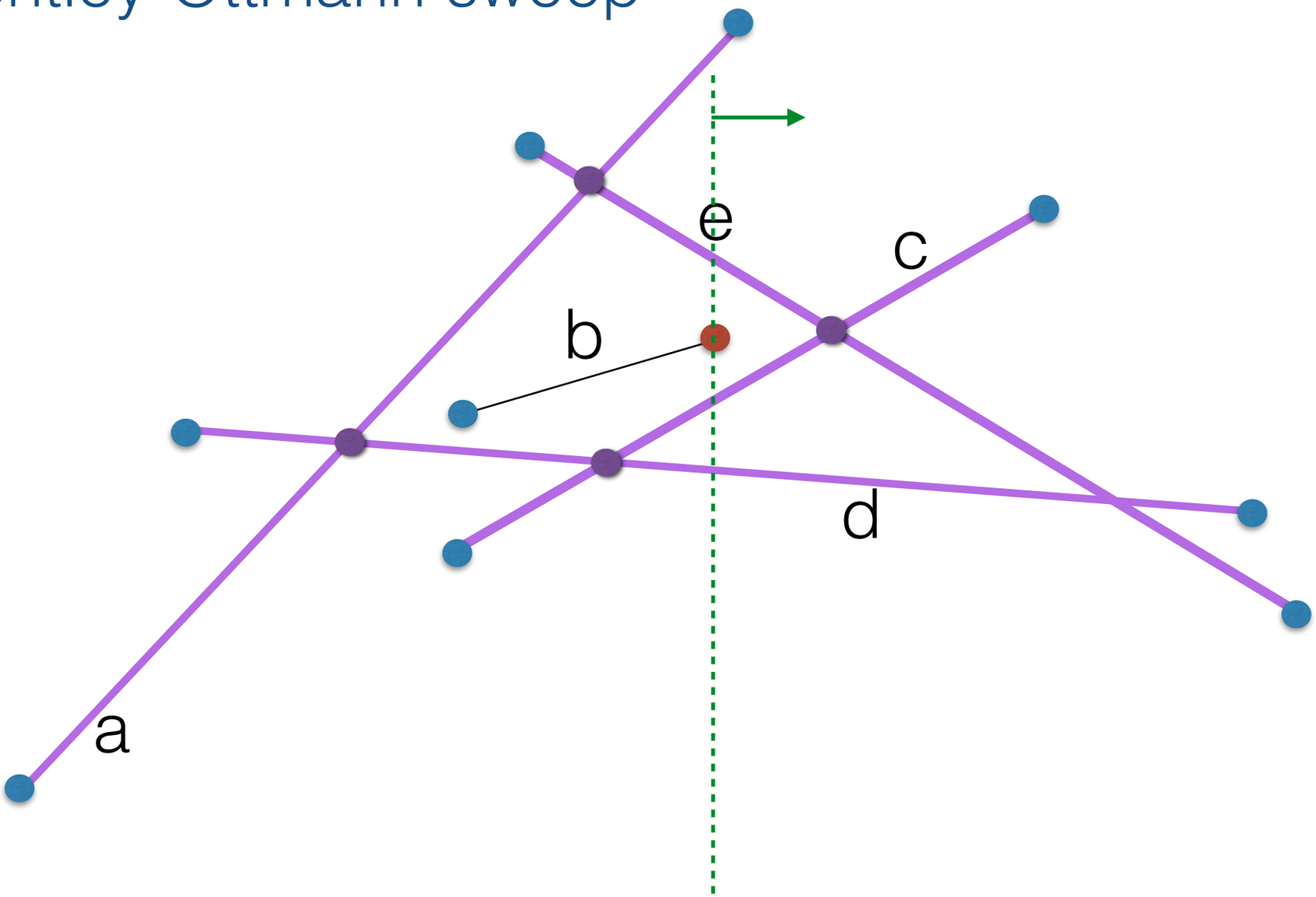


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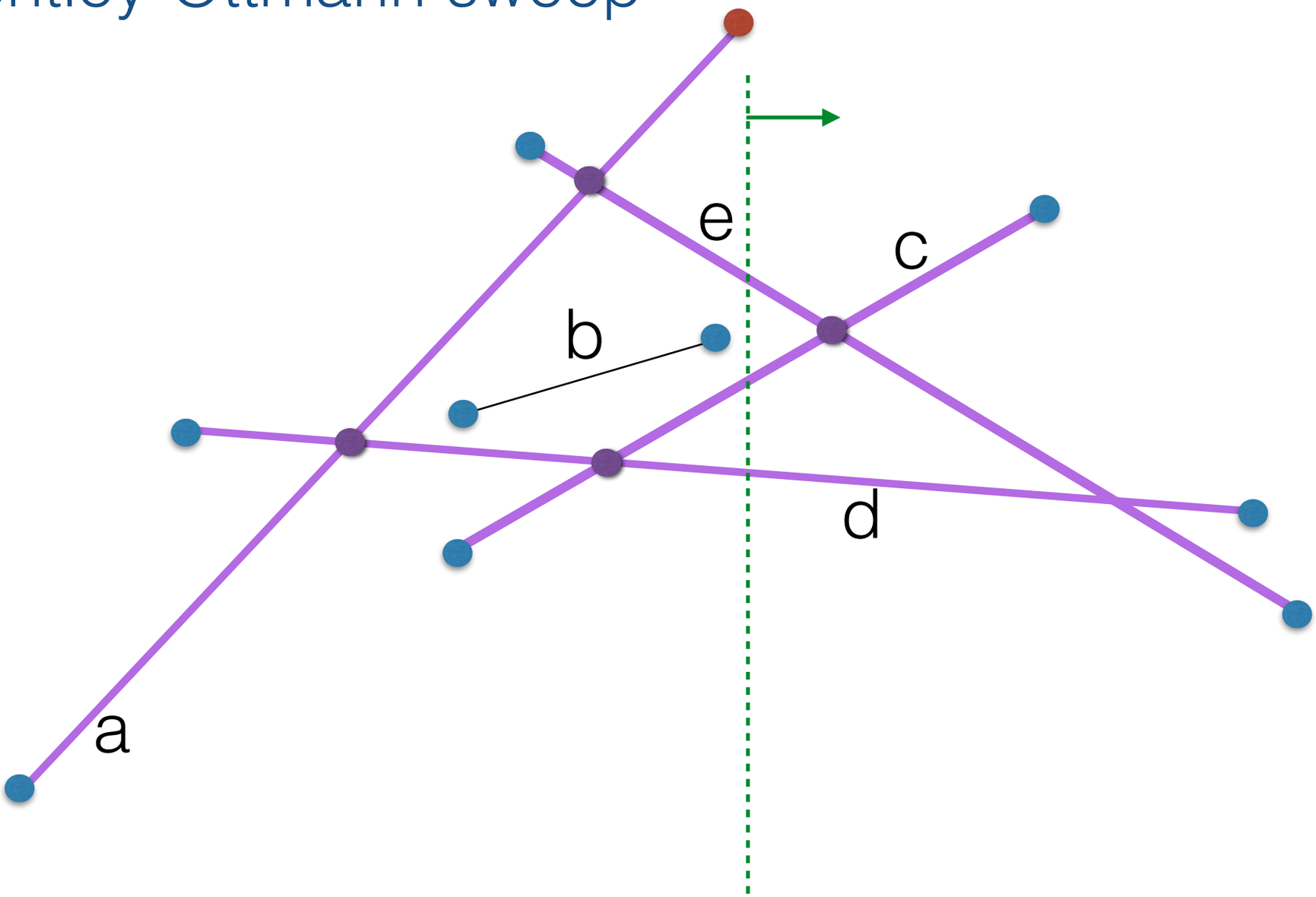
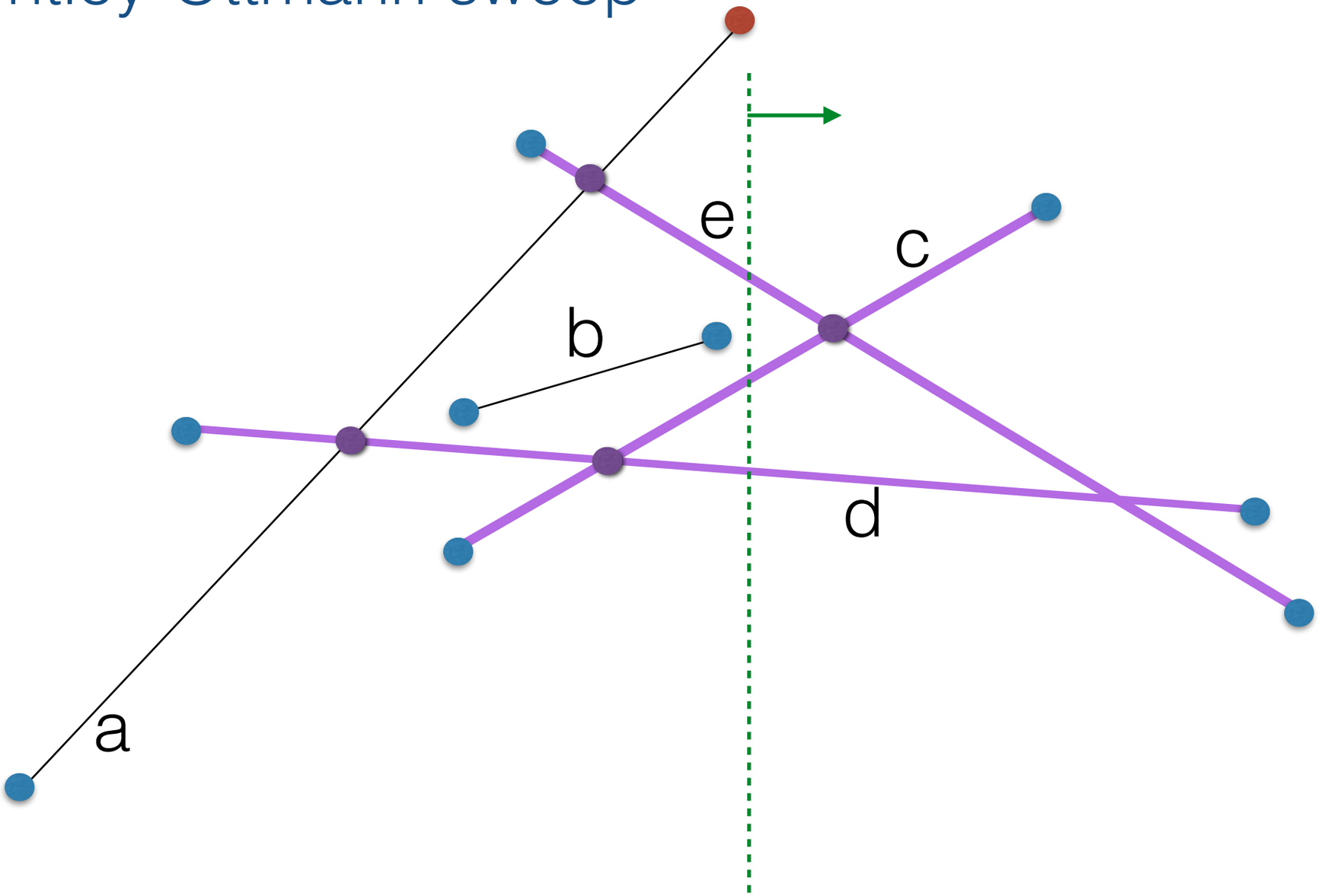



- this event is end of segment a:
  - delete a from AS: d < c < e
  - no new neighbors

# Bentley-Ottmann sweep

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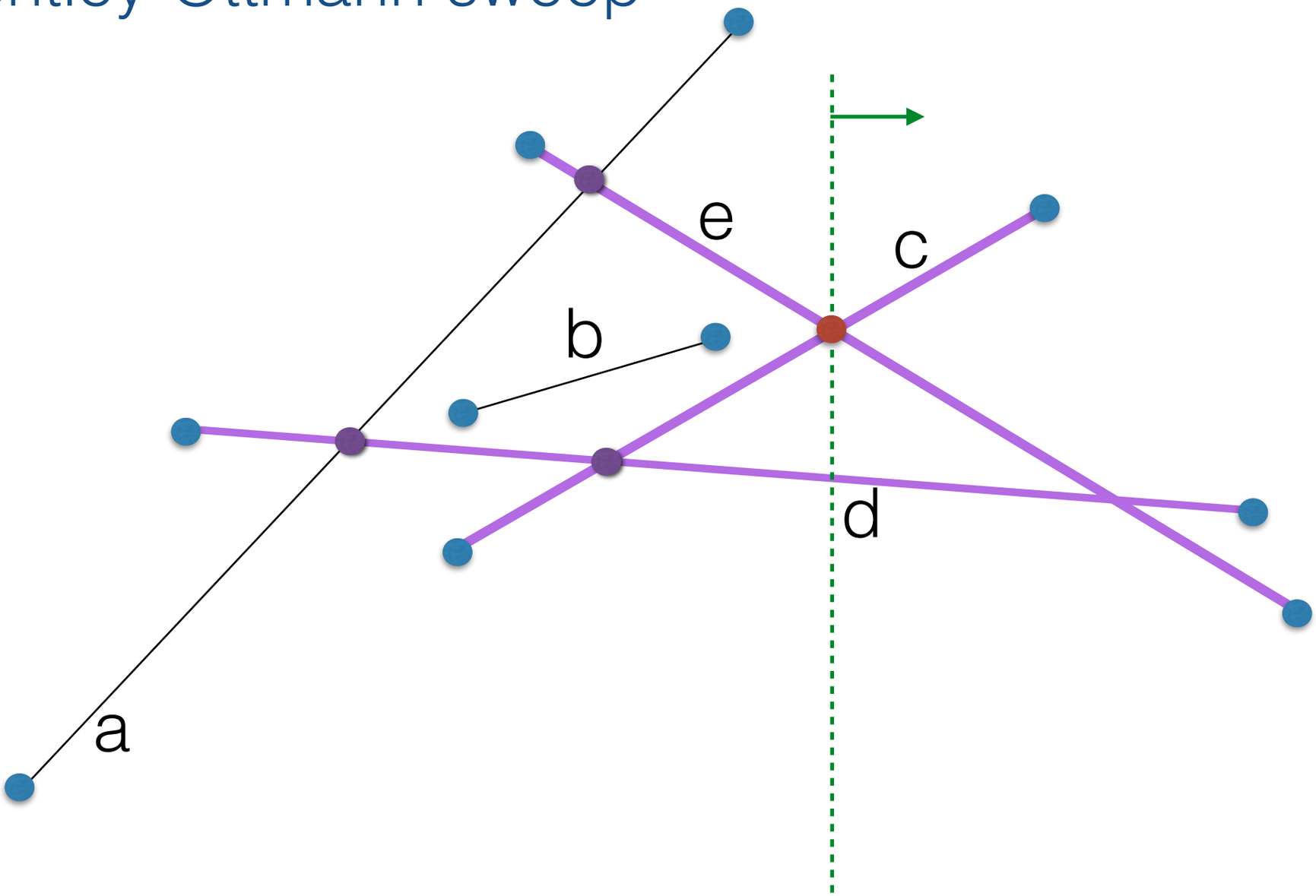
  - delete a from AS: d < c < e
  - no new neighbors

# Bentley-Ottmann sweep




- this event is the intersection of (c,e):
  - flip c,e in AS: d < e < c
  - check new neighbors (d,e) for intersection to the right of the sweep line; this detects the intersection of (d,e); report it and insert it as future event

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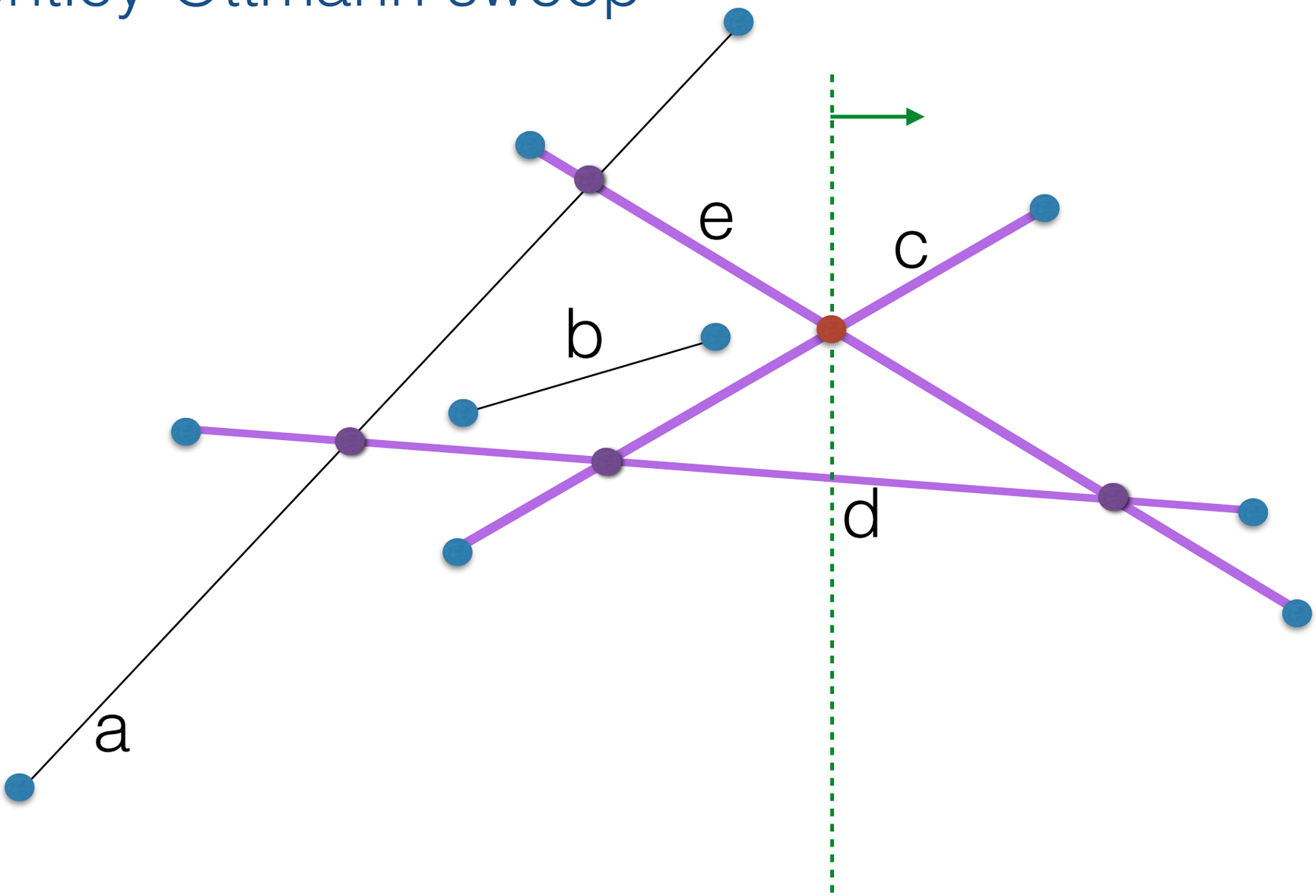


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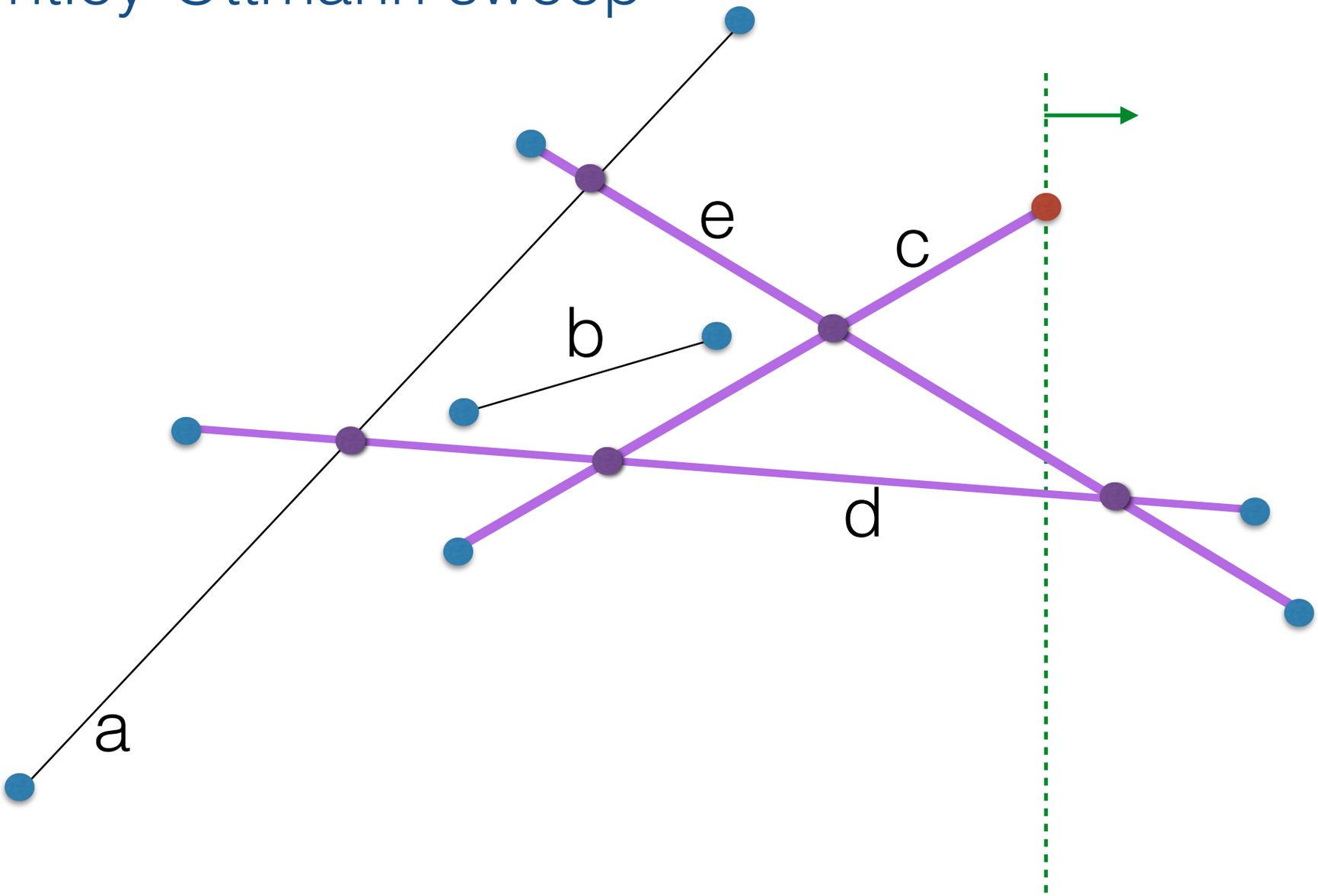
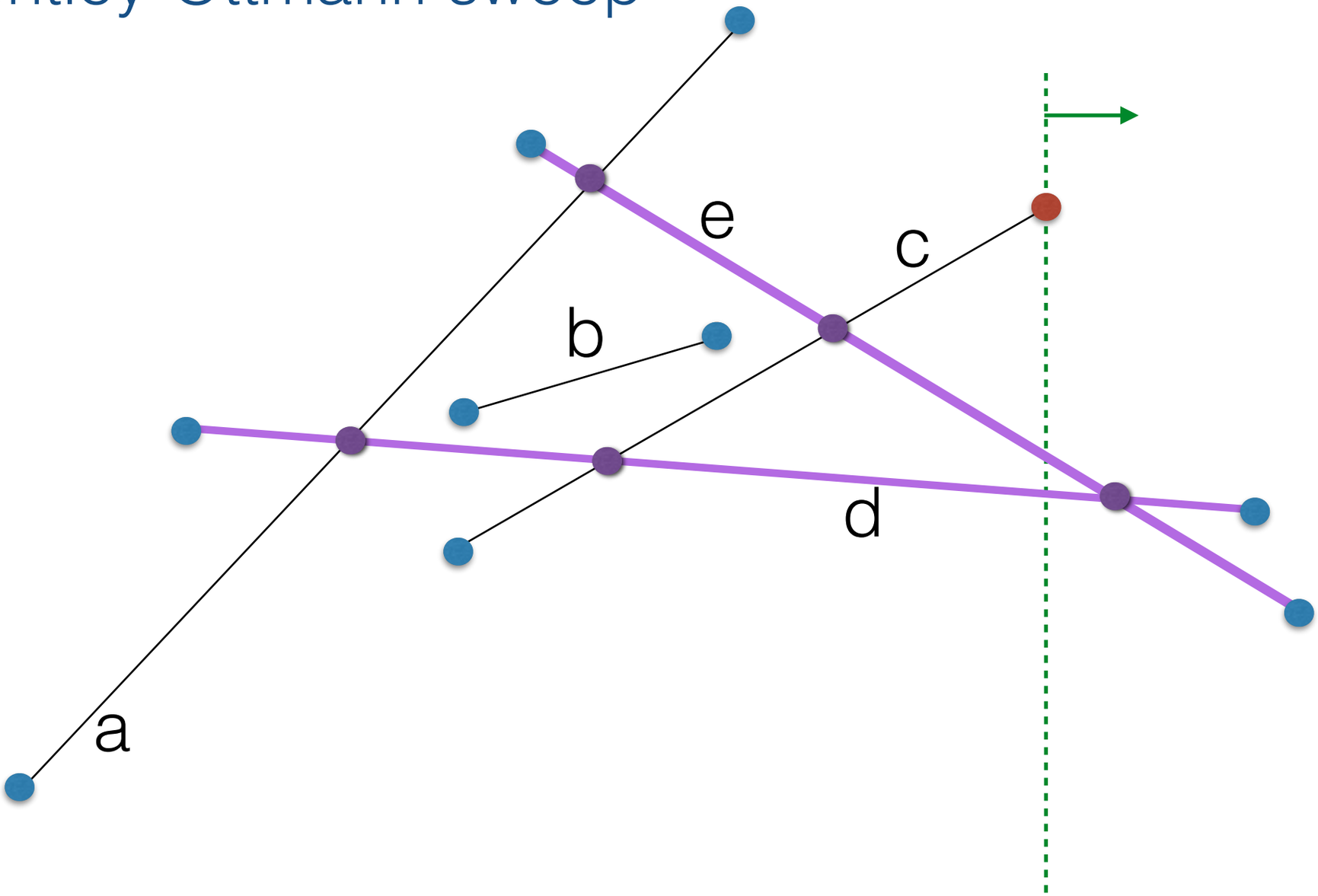



- this event is end of segment c:
  - delete c in AS: d < e
  - no new neighbors

# Bentley-Ottmann sweep

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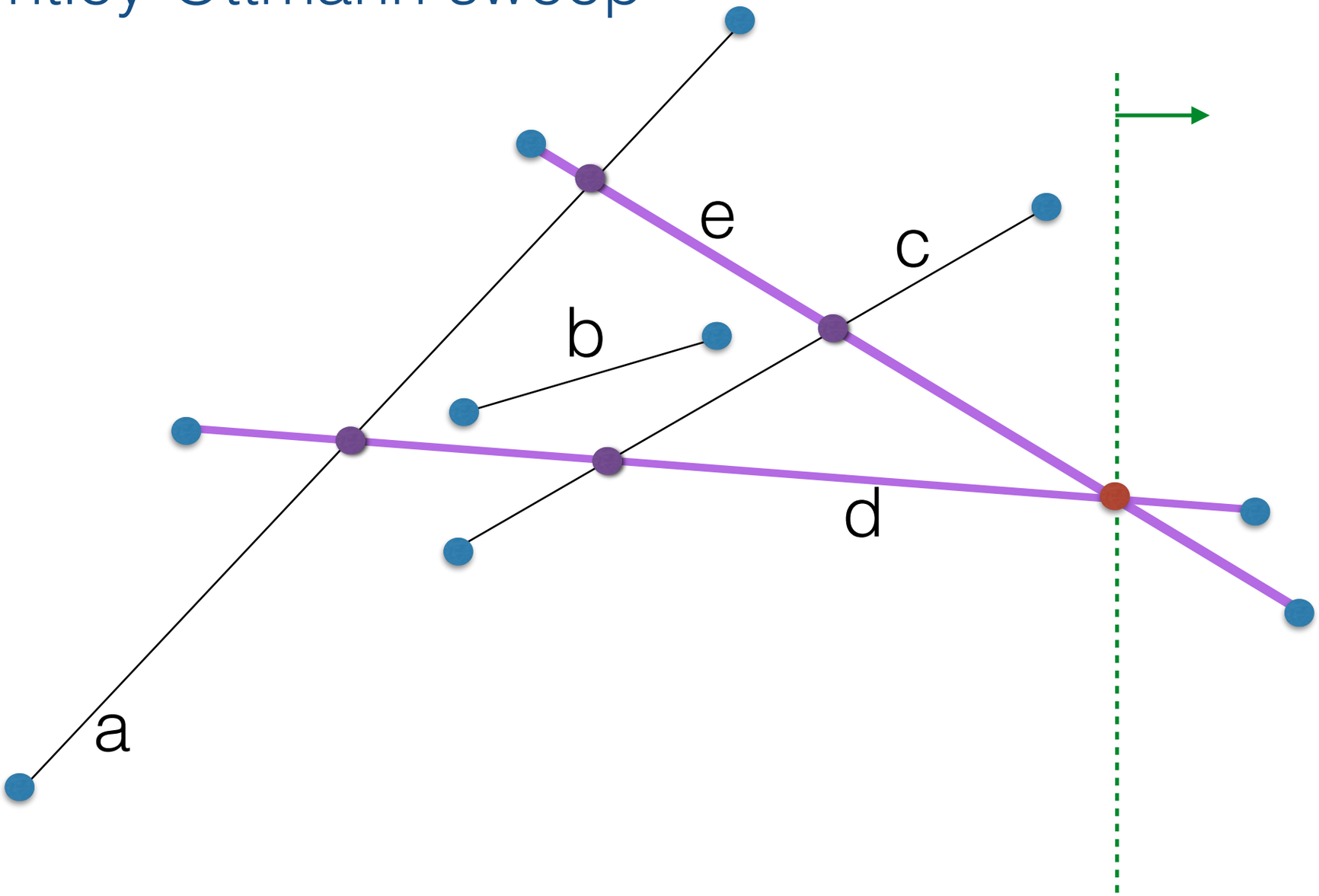


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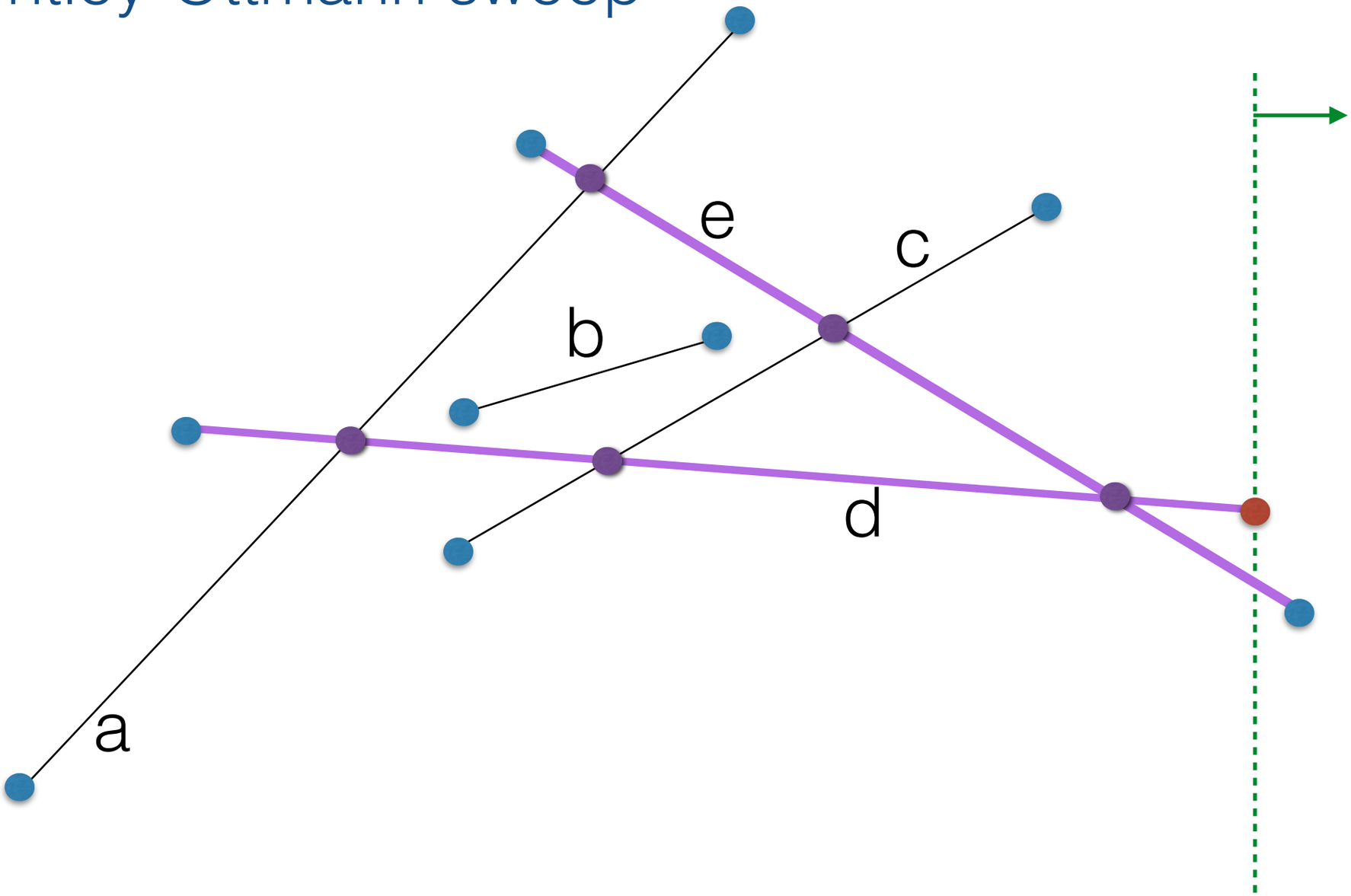


- this event is the end of d:
  - delete d in AS: e
  - no new neighbors

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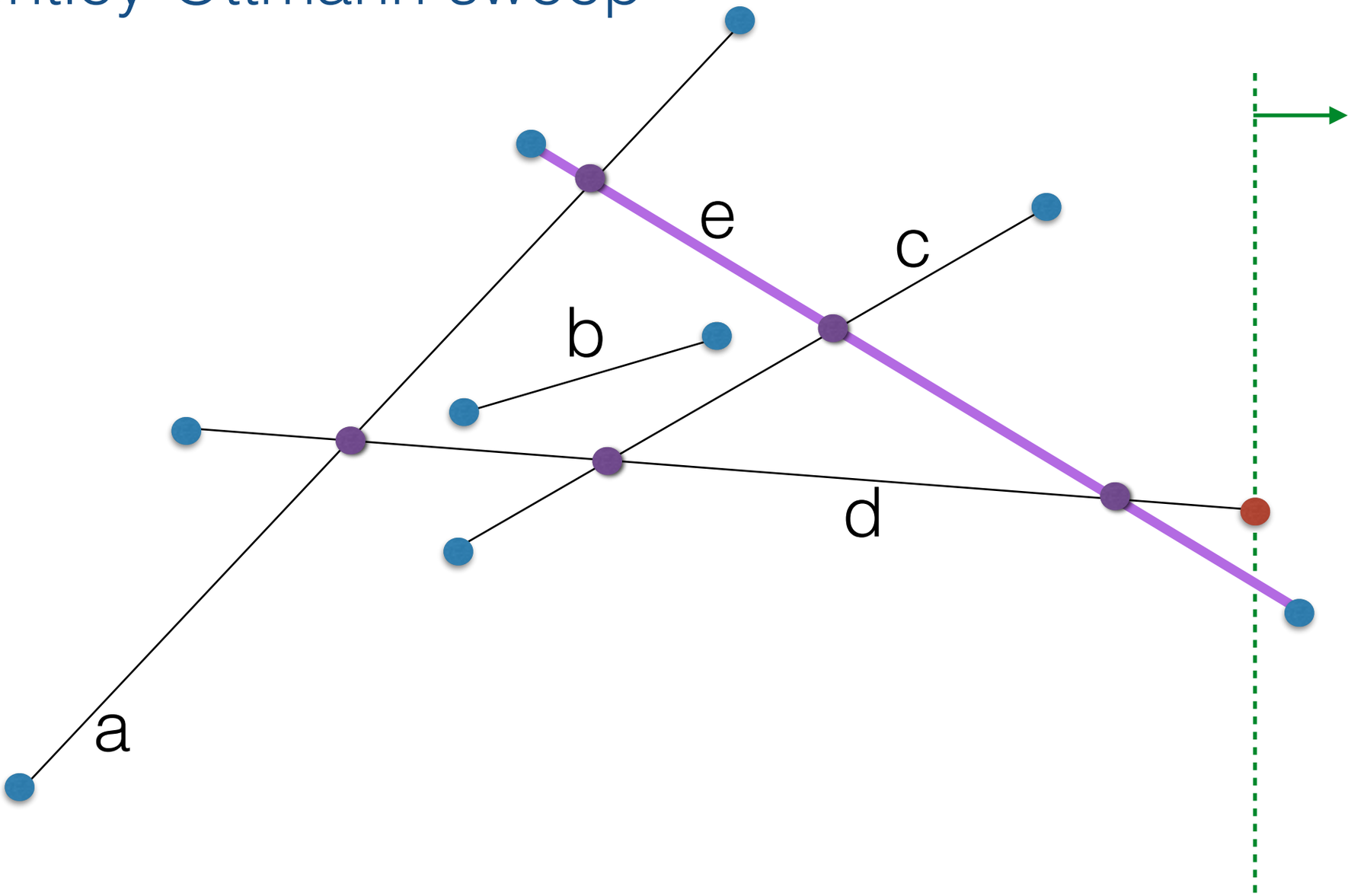


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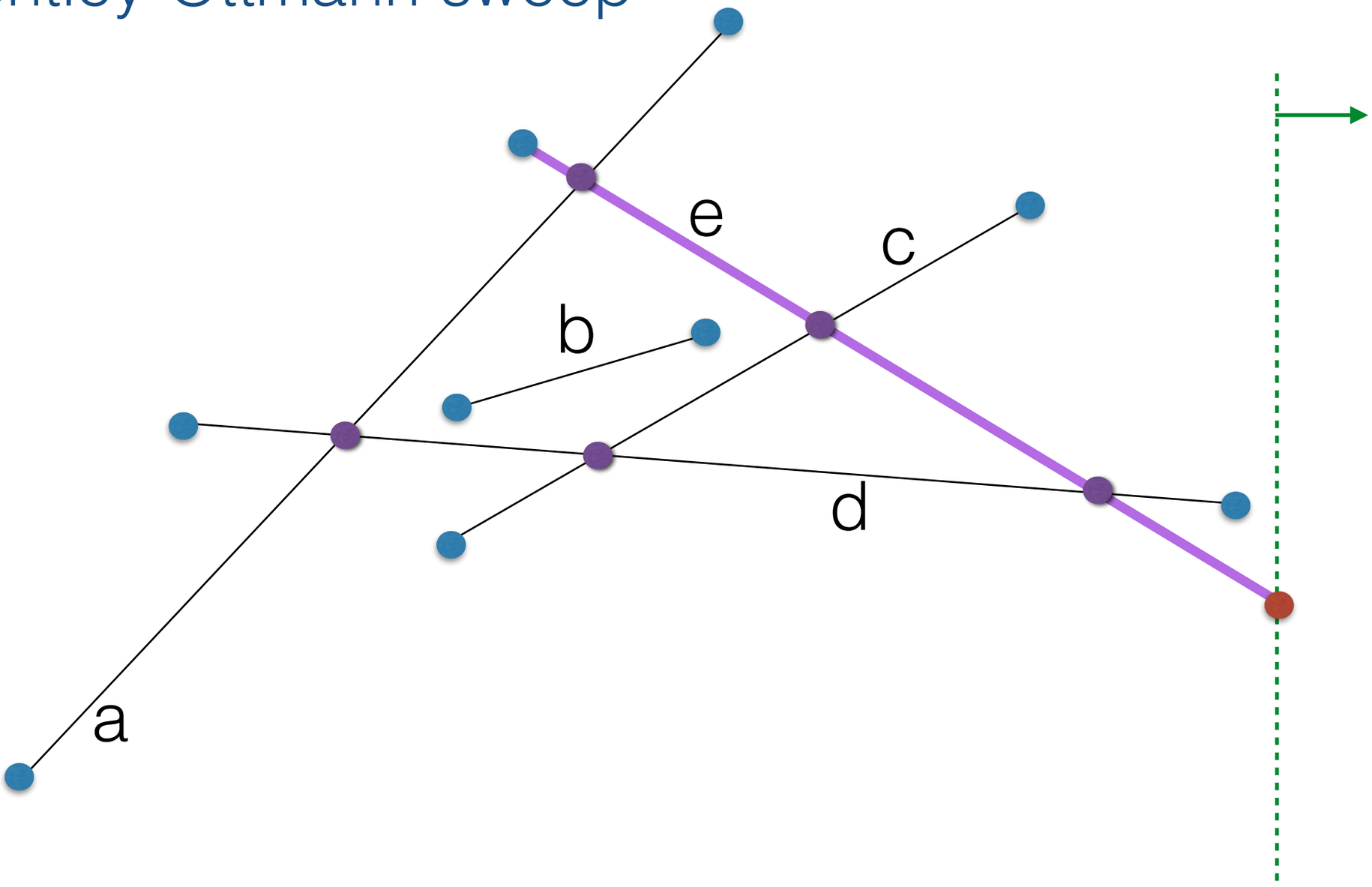


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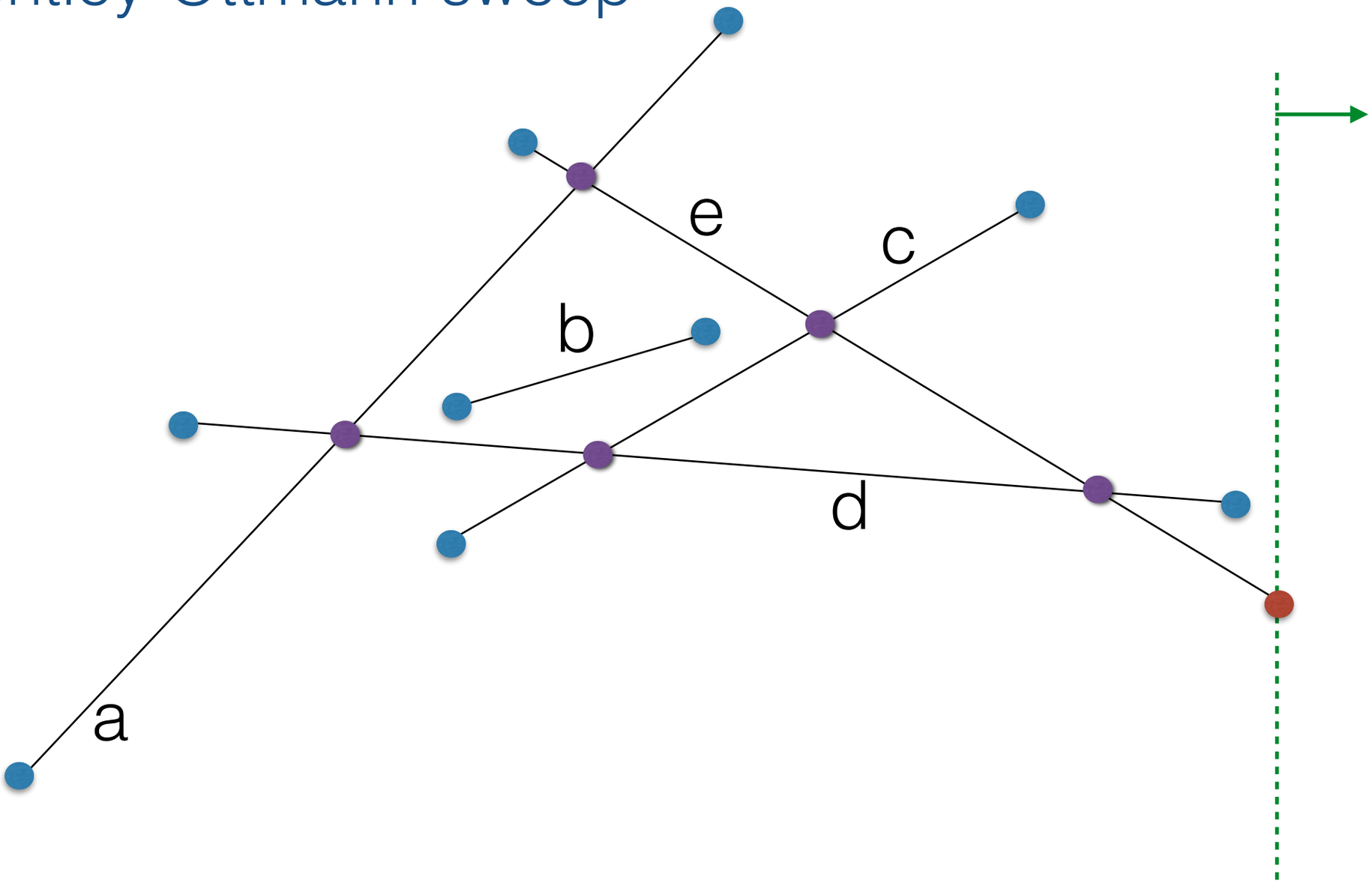


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# Bentley-Ottmann sweep

- Simplifying assumptions
  - no vertical segments
  - no two segments intersect at their endpoints
  - no three (or more) segments have a common intersection
  - all endpoints (of segments) and all intersection points have different x-coordinates
  - no segments overlap

- These assumptions are not realistic for real data..
- But, they don't provide insight into the plane sweep technique, so we omit them
- Real data challenges
  - dealing with degenerate cases
  - dealing with finite precision arithmetic and precision problems

# Bentley-Ottmann sweep

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- Event list EL:
  - For any position of SL, EL contains segment endpoints to the right of SL, and also the intersections to the right of SL of active segments that were/ are neighbors in SL
  - EL is sorted by x-coordinate
- For any position of the sweep line SL, all pairs of intersecting dead segments have been reported.

Algorithm Bentley-Ottmann (S)

//S is a set of n line segments in the plane

- initialize AS= {}
- sort 2n endpoints of all segments in S by x-coord and store them in EventList
- while EventList not empty

    - let e be the next event from EventList; delete it from EL

    //sweep line moves to x=e.x

    - if e is left endpoint of a segment l

        //l becomes active

        - insert l in AS in the right place
        - check if l intersects with l->prev and l->succ in AS to the right of the sweep line; if they do, insert their intersection point in the EventList

        //optional: since l.prev and l.succ are not neighbors anymore, we check if they intersect and if they do, delete that intersection point from the EventList

    - if e is the right endpoint of a segment
        - …

    - if e is the intersection of two segments
        - …
- end.

# Bentley-Ottmann sweep

## Questions

- AS
  - What operations do we do on AS?
  - What data structure should we use for AS?

- EL
  - Note that we know a priori the 2n events corresponding to start and end-points of segments, but EL is not static; the events corresponding to intersection points are generated on the fly
  - What operations do we do on EL?
  - What data structure should we use for EL?

# Bentley-Ottmann sweep

Running time

- AS
  - What is the size of  AS?
    - O(n)
  - How many operations?
    - O(n+k)
  - Overall time?
    - O((n+k)lg n)

- EventList
  - What is the size of EventList?
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  - How many operations?
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Bentley-Ottmann sweep runs in O((n+k) lg n) time.