Computational Geometry
[csci 3250]

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Bowdoin College

1

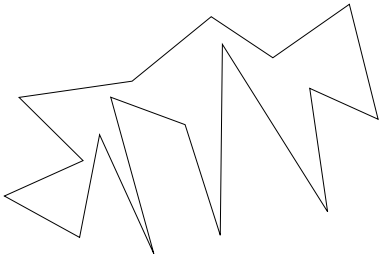
Polygon Triangulation

2

Polygon Triangulation

The problem: Triangulate a given polygon.

(output a set of diagonals that partition the polygon into triangles).



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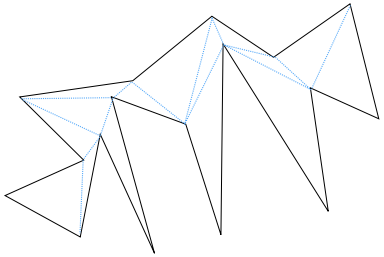

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Definitions

Given a simple polygon P

- A **diagonal** is a segment between 2 non-adjacent vertices that lies entirely within the interior of the polygon.
- A **ear** with tip v_i is a set of 3 consecutive vertices v_{i-1}, v_i, v_{i+1} if $v_{i-1}v_{i+1}$ is a diagonal.
  - Put differently, v_i is an ear tip if the vertex right before it and the vertex right after it are *visible* to each other

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Known Results

- Theorem: Any simple polygon must have a convex vertex.
- Theorem: Any simple polygon with n>3 vertices contains (at least) a diagonal.
- Theorem: Any polygon can be triangulated by adding diagonals.
- Theorem: Any simple polygon has at least two ears.
- All triangulations of a polygon of n vertices have n-2 triangles and n-3 diagonals.

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Algorithms

- Is $p_i p_j$ a diagonal of P?
- Find a diagonal of P
- Is v_i the tip of an ear?
- Find all ears

==> Triangulate by finding ears

Polygon triangulation: First steps

- **Algorithm 1:** Triangulation by identifying ears
  - Idea: Find an ear, output the diagonal, delete the ear tip, repeat.
  - Analysis:
    - checking whether a vertex is ear tip or not: O(n)
    - checking all vertices: $O(n^2)$
    - overall $O(n^3)$
- **Algorithm 2:** Triangulation by finding diagonals
  - Idea: Find a diagonal, output it, recurse.
  - A diagonal can be found in O(n) time (using the proof that a diagonal exists)
  - $O(n^2)$

Polygon triangulation: First steps

- **Algorithm 3**: Triangulation by identifying ears in $O(n^2)$
  - Find an ear, output the diagonal, delete the ear tip, repeat.
  - Avoid recomputing ear status for all vertices every time
    - When you remove a ear tip from the polygon, which vertices might change their ear status?

History of Polygon Triangulation

- Early algorithms: $O(n^4)$, $O(n^3)$, $O(n^2)$
- Several O(n lg n) algorithms known ← practical
- …
- Many papers with improved bounds ← not practical
- …
- 1991: Bernard Chazelle (Princeton) gave an O(n) algorithm ← not practical
  - https://www.cs.princeton.edu/~chazelle/pubs/polygon-triang.pdf
  - Ridiculously complicated, not practical
  - O(1) people actually understand it (and I'm not one of them)
- No algorithm is known that is practical enough to run faster than the O(n lg n) algorithms
- OPEN problem
  - A practical algorithm that's better than O(n lg n).

An O(n lg n) Polygon Triangulation Algorithm

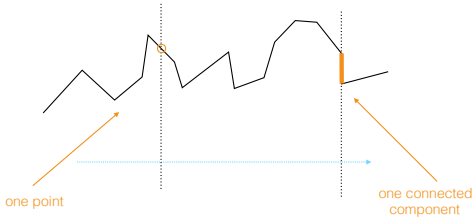
- Ingredients
  - Consider the special case of triangulating a **monotone**/**unimonotone** polygon
  - Convert an arbitrary polygon into monotone/unimonotone polygons

Monotone chains

A polygonal chain is **x-monotone** if any line perpendicular to x-axis intersects it in one point (one connected component).

Monotone chains

A polygonal chain is **x-monotone** if any line perpendicular to **x-axis** intersects it in one point (one connected component).



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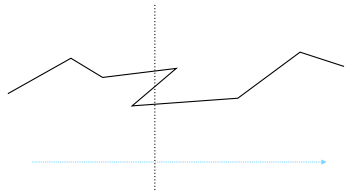
Monotone chains



Not x-monotone

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Monotone chains

- Claim: Let u and v be the points on the chain with min/max x-coordinate. The vertices on the boundary of an x-monotone chain, going from u to v, are in x-order.

x-monotone

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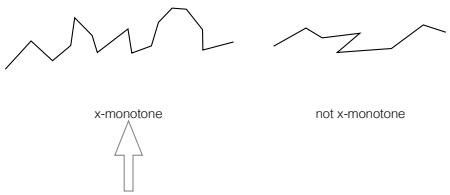
Monotone chains



x-monotone

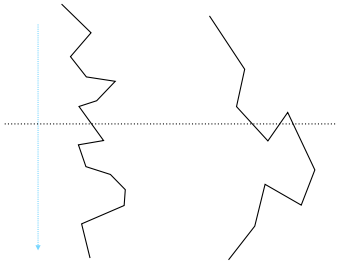
not x-monotone

As you travel along this chain, your x-coordinate is staying the same or increasing

16

Monotone chains

A polygonal chain is **y-monotone** if any line perpendicular to **y-axis** intersects it in one point (one connected component).



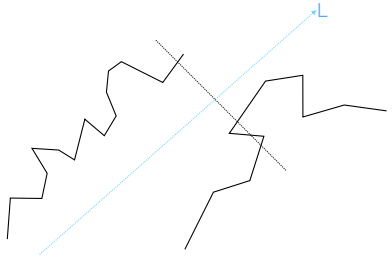
y-monotone

not y-monotone

17

Monotone chains

A polygonal chain is **L-monotone** if any line perpendicular to **line L** intersects it in one point (one connected component).



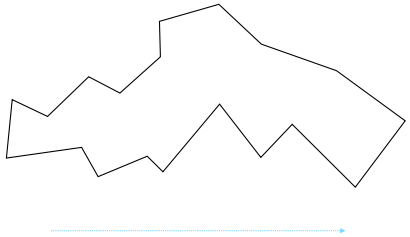
y-monotone

not L-monotone

18

Monotone polygons

A polygon is **x-monotone** if its boundary can be split into two x-monotone chains.



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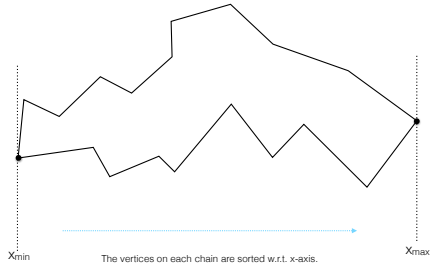
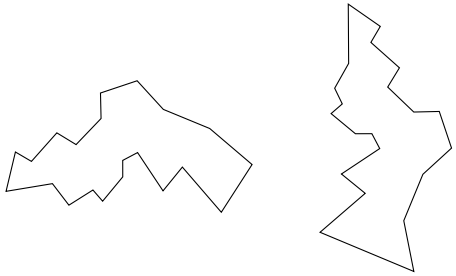


The vertices on each chain are sorted w.r.t. x-axis.

20

Monotone polygons



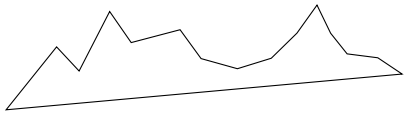
x-monotone

y-monotone

21

Monotone Mountains

A polygon is an **x-monotone mountain** if it is monotone and one of the two chains is a single segment.



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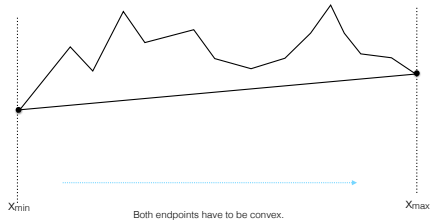

Both endpoints have to be convex.

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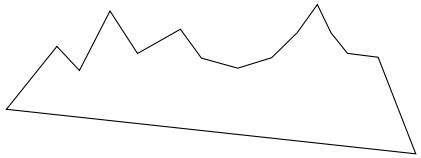
Monotone mountains are easy to triangulate!



Class work: come up with an algorithm and analyze it.

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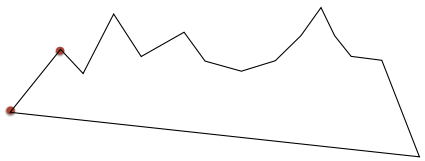

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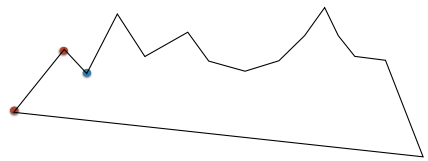

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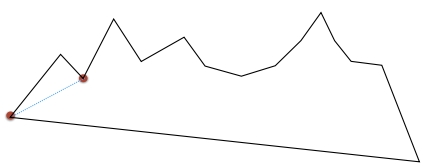

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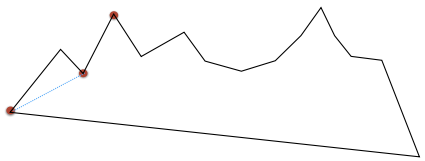

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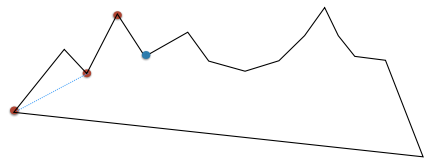

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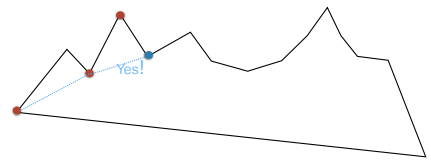

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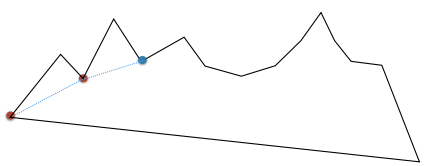

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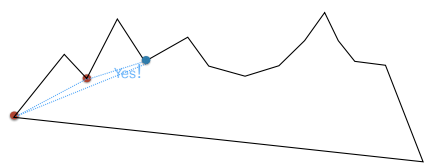

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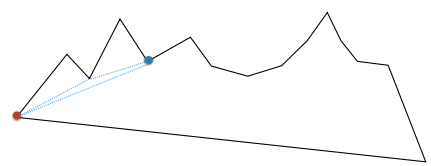

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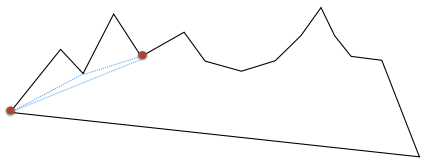

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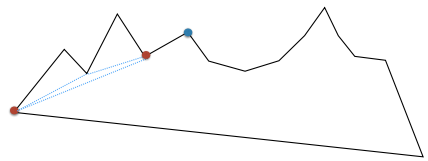

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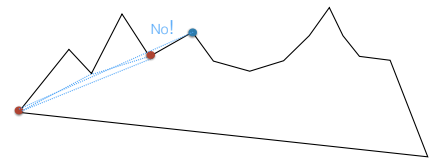

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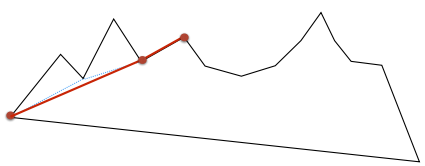

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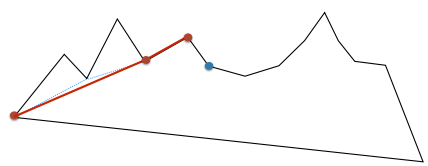

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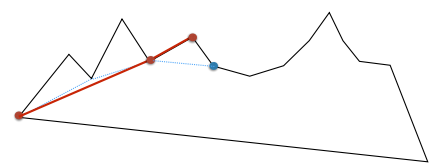

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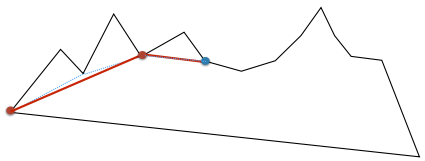

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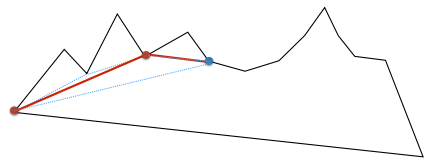

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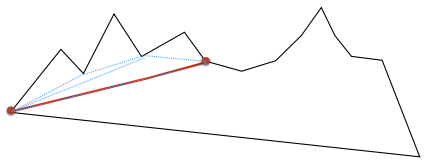

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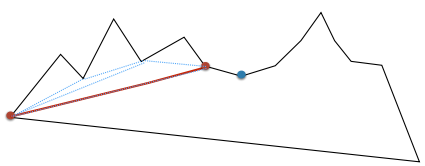

46

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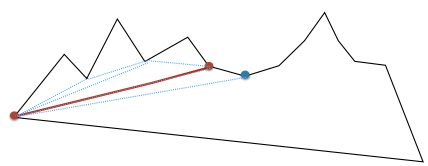

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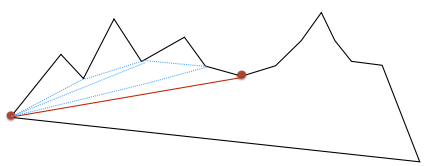
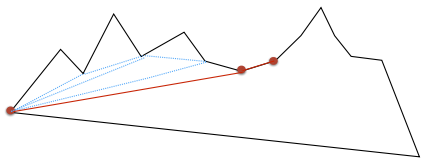


48

Monotone mountains are easy to triangulate!



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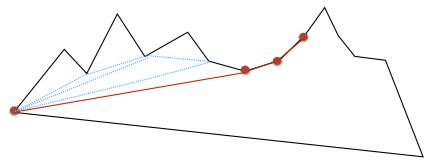

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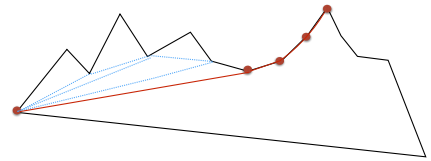

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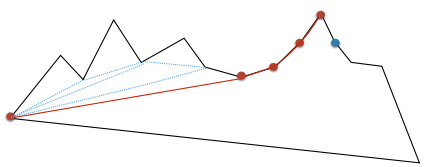

52

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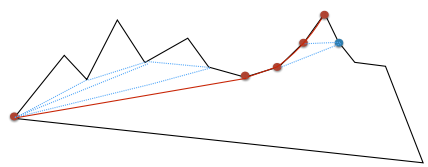

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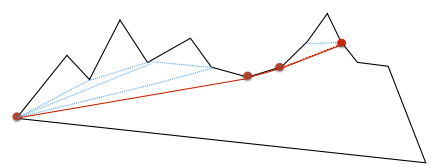

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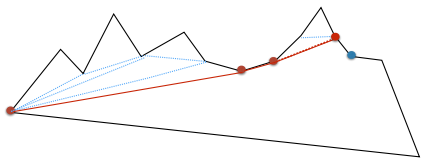

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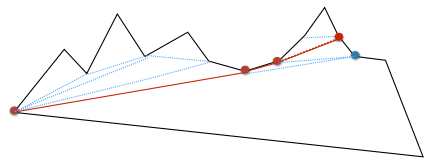

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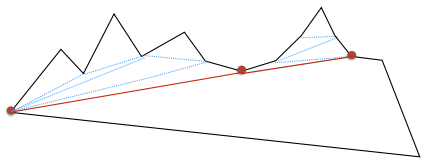

57

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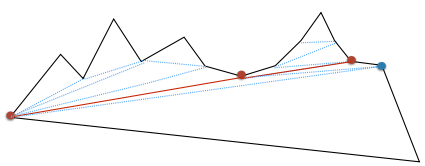

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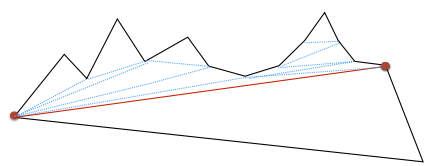

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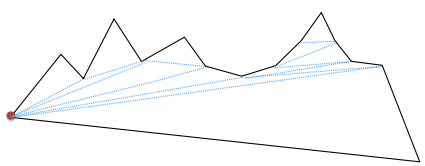

Analysis: O(n) time

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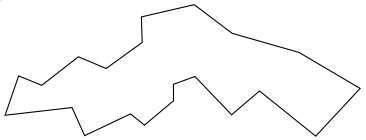
Triangulating Monotone Polygons

Similar idea, pick the next vertex in x-order. It can be on upper or lower chain.

O(n) time



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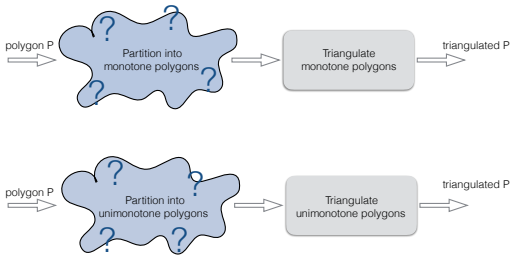
Towards an O(n lg n) Polygon Triangulation Algorithm

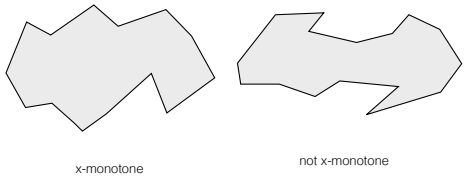


62

How can we partition a polygon into (uni)monotone pieces?

63

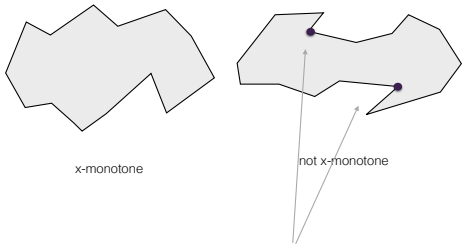
Intuition



What makes a polygon **not** monotone?

64

Intuition

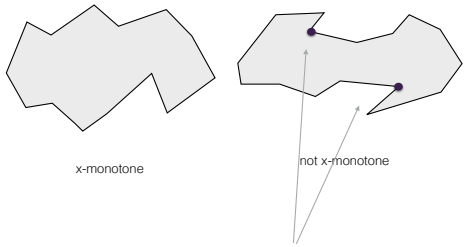


What makes a polygon **not** monotone?

65

Intuition

Cusp: a reflex vertex v such that the vertices before and after are both smaller or both larger than v (in terms of x-coords).

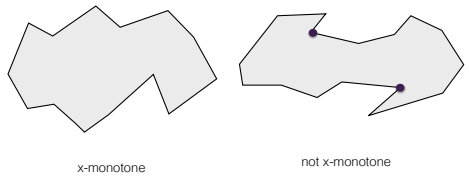


What makes a polygon **not** monotone?

66

Intuition

Cusp: a reflex vertex v such that the vertices before and after are both smaller or both larger than v (in terms of x-coords).



- Theorem: If a polygon has no cusps, then it's monotone.
- Proof: Intuitively clear, but proof a little tedious. Maybe later..

67

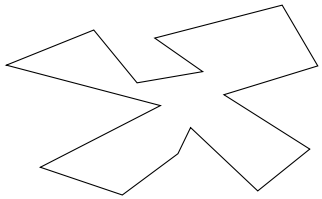
We'll get rid of cusps using a trapezoidation of P.

68

Trapezoid partitions

Shoot vertical rays

- If polygon is above vertex, shoot vertical ray up until reaches boundary
- If f polygon is below vertex, shoot down
- If polygon is above and below vertex, shoot both up and down



69

Trapezoid partitions

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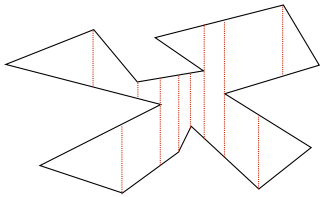
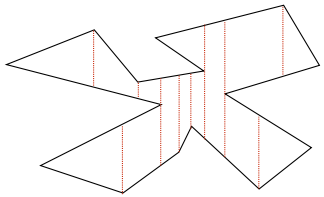


70

Trapezoid partitions

Properties

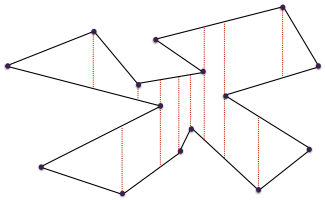
- Each polygon in the partition is a trapezoid, because:
  - It has one or two threads as sides.
  - If it has two, then they must both hit the same edge above, and the same edge below.
- At most one thread through each vertex => O(n) threads => O(n) trapezoids



71

Trapezoid partitions

- Each trapezoid has precisely two vertices of the polygon, one on the left and one on the right. They can be on the top, bottom or middle of the trapezoid.



72

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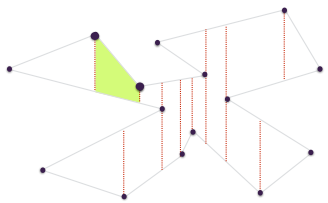

73

Trapezoid partitions

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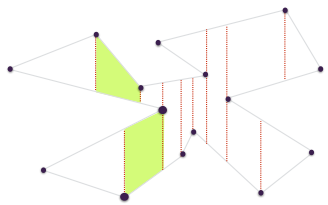

74

Trapezoid partitions

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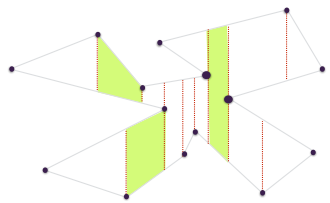
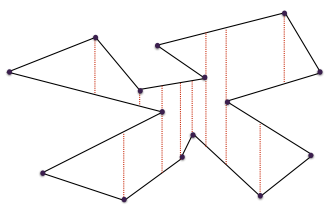


75

Diagonals

- In each trapezoid: if its two vertices are not on the same edge, they define a **diagonal**.



76

Diagonals

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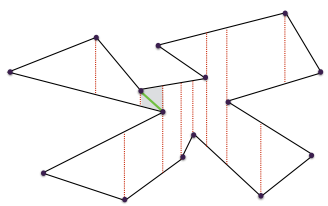

77

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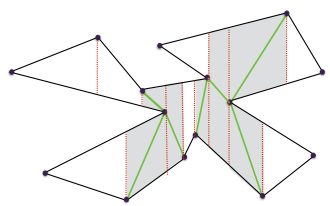
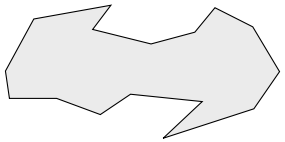


78

We can use the trapezoid partition of P to “split” the cusps

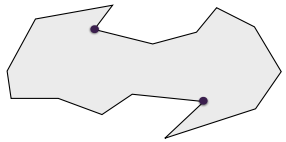
79

Removing cusps



80

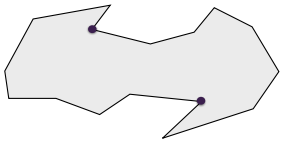
Removing cusps



1. Identify cusp vertices

81

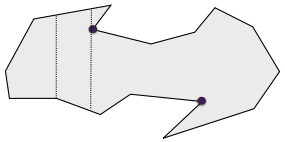
Removing cusps



1. Identify cusp vertices
2. Compute a trapezoid partition of P
3. For each cusp vertex, add diagonal in trapezoid before/after the cusp

82

Removing cusps



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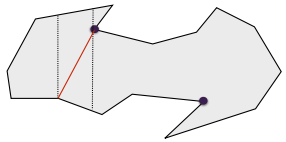
83

Removing cusps



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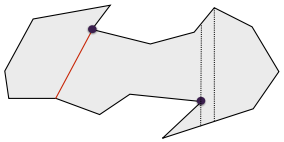
84

Removing cusps



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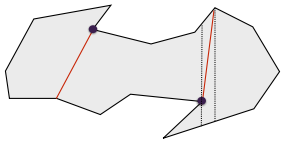
85

Removing cusps



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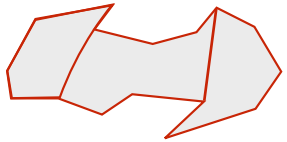
86

Removing cusps

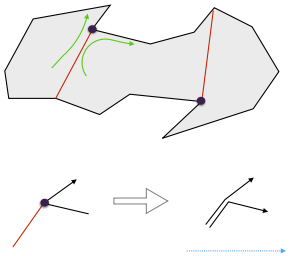


1. Identify cusp vertices
2. Compute a trapezoid partition of P
3. For each cusp vertex, add diagonal in trapezoid before/after the cusp

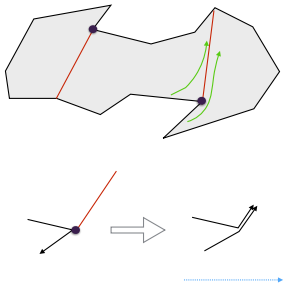
This creates a partition of P.

Claim: The resulting polygons have no cusps and thus are monotone (by theorem).

87



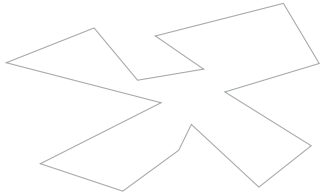
88



89

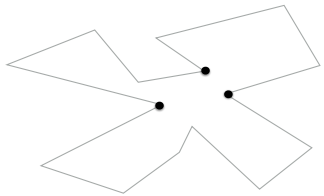
Removing cusps

- Another example



90

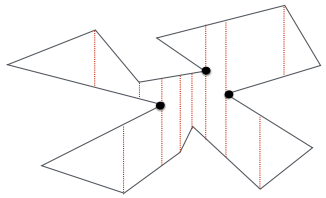
Removing cusps



1. Identify cusp vertices

91

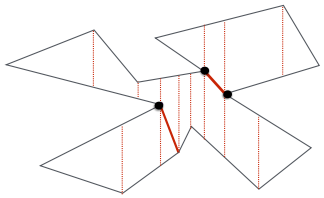
Removing cusps



1. Identify cusp vertices
2. Compute a trapezoid partition of P

92

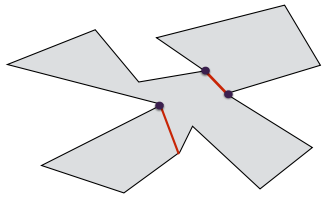
Removing cusps



1. Identify cusp vertices
2. Compute a trapezoid partition of P
3. Add obvious diagonal before/after each cusp

93

Removing cusps

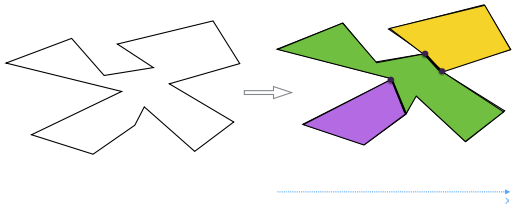


This partitions the polygon into monotone pieces.

94

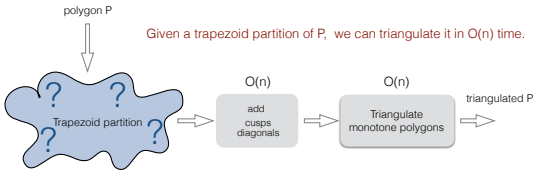
Partition P into monotone polygons

1. Identify cusp vertices
2. Compute a trapezoid partition of P
3. Add obvious diagonal before/after each cusp



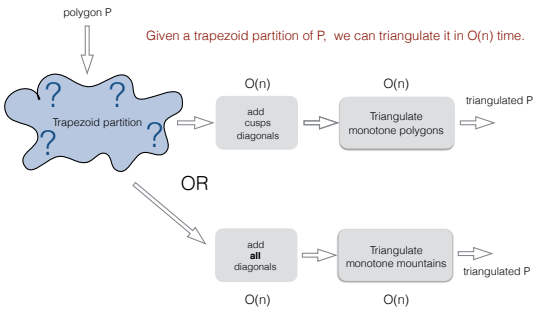
95

Towards an O(n lg n) Polygon Triangulation Algorithm



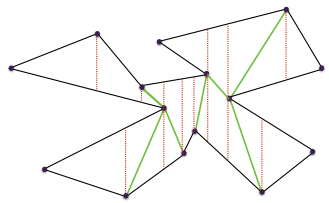
96

Towards an O(n lg n) Polygon Triangulation Algorithm



97

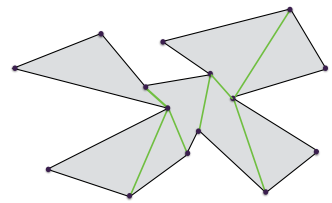
Partitioning into monotone mountains



1. Compute a trapezoid partition of P
2. Output **all** diagonals.

98

Partitioning into monotone mountains

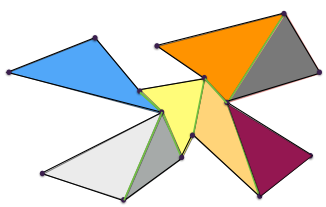


1. Compute a trapezoid partition of P
2. Output **all** diagonals.

   Claim: All diagonals partition the polygon into monotone mountains.

99

Partitioning into monotone mountains

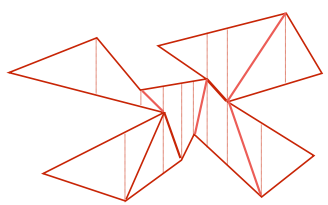


1. Compute a trapezoid partition of P
2. Output **all** diagonals.

   Claim: The diagonals partition the polygon into monotone mountains.

100

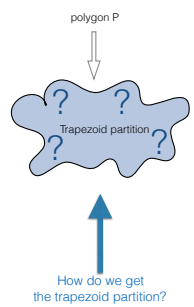
Claim: The diagonals partition the polygon into monotone mountains.



Proof:

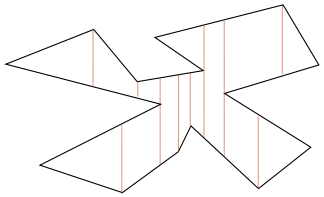
- Each polygon is monotone
- One of the chains must be a segment (because if it had another point, that point would generate a diagonal)

101



102

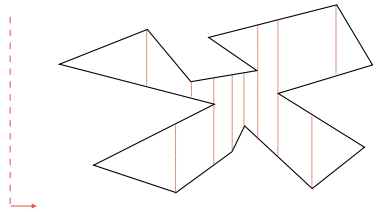
Computing the trapezoid partition



103

Computing the trapezoid partition in O(n lg n)

- Plane sweep



104

Computing the trapezoid partition in O(n lg n)

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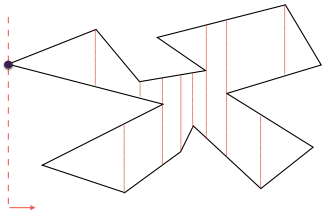

105

Computing the trapezoid partition in O(n lg n)

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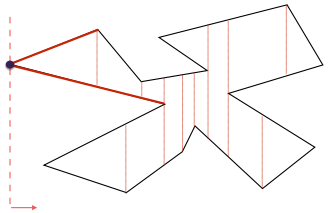
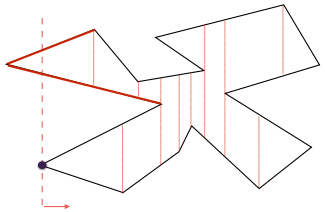


106

Computing the trapezoid partition in O(n lg n)

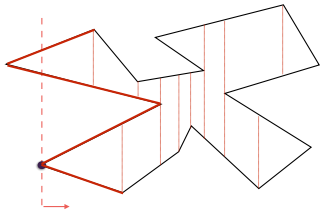
- Plane sweep



107

Computing the trapezoid partition in O(n lg n)

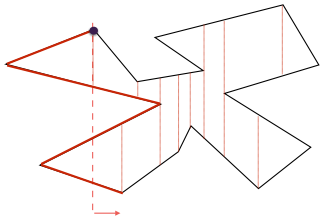
- Plane sweep



108

Computing the trapezoid partition in O(n lg n)

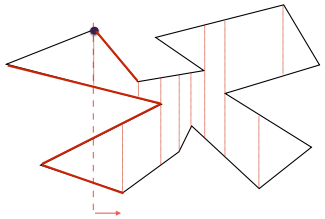
- Plane sweep



109

Computing the trapezoid partition in O(n lg n)

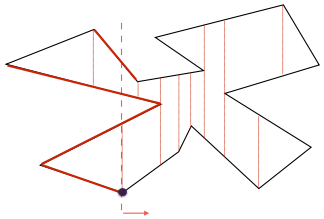
- Plane sweep



110

Computing the trapezoid partition in O(n lg n)

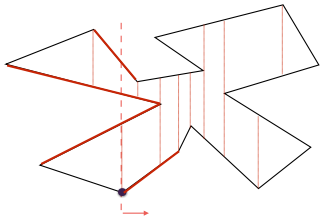
- Plane sweep



111

Computing the trapezoid partition in O(n lg n)

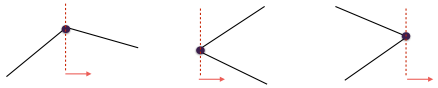
- Plane sweep



112

Computing the trapezoid partition in O(n lg n)

- Plane sweep
- Events: polygon vertices
- Status structure: edges that intersect current sweep line, in y-order
- Events:



How do we determine the trapezoids?

113

Computing the trapezoid partition in O(n lg n)

- Algorithm

114