Imagine an art gallery whose floor plan is a simple polygon, and a guard (a point) inside the gallery.

What does the guard see?

We say that two points $a, b$ are visible if segment $ab$ stays inside $P$ (touching boundary is ok).

The Art Gallery Problem

Imagine an art gallery whose floor plan is a simple polygon, and a guard (a point) inside the gallery.

What does the guard see?

We say that two points $a, b$ are visible if segment $ab$ stays inside $P$ (touching boundary is ok).
We say that a set of guards covers polygon $P$ if every point in $P$ is visible to at least one guard.

**The Art Gallery Problem(s)**

Examples:

1. Does the point guard the triangle?
2. Can all triangles be guarded with one point?
3. Can all quadrilaterals be guarded with one point?

**Questions:**
1. Given a polygon $P$ of size $n$, what is the smallest number of guards (and their locations) to cover $P$?
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The Art Gallery Problem(s)

Questions:
1. Given a polygon $P$ of size $n$, what is the smallest number of guards (and their locations) to cover $P$? NP-Complete.
2. Klee’s problem: Consider all polygons of $n$ vertices, and the smallest number of guards to cover each of them. What is the worst-case?

Klee’s problem:

Notation
- Let $P_n$: polygon of $n$ vertices
- Let $g(P)$: the smallest number of guards to cover $P$
- Let $G(n)$: max $\{g(P)\}$ for all $P_n$.

- $G(n)$ is the smallest number that always works for any $n$-gon. It is sometimes necessary and always sufficient to guard a polygon of $n$ vertices.
- $G(n)$ is necessary: there exists a $P_n$ that requires $G(n)$ guards
- $G(n)$ is sufficient: any $P_n$ can be guarded with $G(n)$ guards.

- Klee’s problem: find $G(n)$

Klee’s problem: find $G(n)$

Our goal (i.e., Klee’s goal) is to find $G(n)$.

Trivial bounds:
- $G(n) \geq 1$: obviously, you need at least one guard.
- $G(n) \leq n$: place one guard in each vertex

$n=3$

Klee’s Problem

$n=3$

Any triangle needs at least one guard.
One guard is always sufficient.

$G(3) = 1$

$n=4$

Klee’s Problem

$n=4$

Any quadrilateral needs at least one guard.
One guard is always sufficient.

$G(4) = 1$
Klee’s Problem

\(n = 5\)

Can all 5-gons be guarded by one point?

\(G(5) = ?\)

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Klee’s Problem

\(n = 5\)

\(G(5) = 1\)

20

Klee’s Problem

\(n = 6\)

A 6-gon that can’t be guarded by one point?

\(G(6) = ?\)

21

Klee’s Problem

\(n = 6\)

\(G(6) = 2\)

22

Klee’s Problem

\(G(n) = ?\)

Come up with a \(P_n\) that requires as many guards as possible.

23

Klee’s Problem

\(G(n) = ?\)

Come up with a \(P_n\) that requires as many guards as possible.

24
Klee's Problem

$G(n) = ?$

Come up with a $P_n$ that requires as many guards as possible.

Klee's Problem

$n/3$ necessary

It was shown that $n/3$ is also sufficient. That is,

Any $P_n$ can be guarded with at most $n/3$ guards.

• (Complex) proof by induction
• Subsequently, simple and beautiful proof due to Steve Fisk, who was Bowdoin Math faculty.
• Proof in The Book.

Fisk's proof of sufficiency

1. Any simple polygon can be triangulated.
2. A triangulated simple polygon can be 3-colored.
3. Observe that placing the guards at all the vertices assigned to one color guarantees the polygon is covered.
4. There must exist a color that's used at most $n/3$ times. Pick that color and place guards at the vertices of that color.

Fisk's proof of sufficiency

Claim: Any simple polygon can be triangulated.
Given a simple polygon P, a diagonal is a segment between 2 non-adjacent vertices that lies entirely within the interior of the polygon.

Claim: Any simple polygon can be triangulated.
Proof idea: induction using the existence of a diagonal. Later.

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Fisk's proof of sufficiency

1. Any simple polygon can be triangulated
2. Any triangulation of a simple polygon can be 3-colored.

• Placing guards at vertices of one color covers $P$.
• Pick least frequent color! At most $n/3$ vertices of that color.

The proofs
Fisk's proof of sufficiency

1. Any polygon can be triangulated.
2. Any triangulation can be 3-colored.
3. Observe that placing the guards at all the vertices assigned to one color guarantees the polygon is covered.
4. There must exist a color that's used at most n/3 times. Pick that color and place guards at the vertices of that color.

Claim: The set of red vertices covers the polygon. The set of blue vertices covers the polygon. The set of green vertices covers the polygon.

Proof:

There are n vertices colored with one of 3 colors.

Claim: There must exist a color that's used at most n/3 times.

Proof:

Theorem: Any triangulation can be 3-colored.

Proof:

Theorem: Any simple polygon with n > 3 vertices contains (at least) a diagonal.

Proof:
Theorem: Any polygon can be triangulated

Proof: