Dynamic Programming

(CLRS 15.2-15.3)

Today we discuss a technique called "Dynamic programming". It is neither especially 'dynamic' nor especially 'programming' related. We will discuss dynamic programming by looking at an example.

1 Matrix-chain multiplication

- Problem: Given a sequence of matrices $A_1, A_2, A_3, ..., A_n$, find the best way (using the minimal number of multiplications) to compute their product.
  - Isn't there only one way? $((\cdots((A_1 \cdot A_2) \cdot A_3) \cdots) \cdot A_n)$
  - No, matrix multiplication is *associative*.
    e.g. $A_1 \cdot (A_2 \cdot (A_3 \cdot (\cdots (A_{n-1} \cdot A_n) \cdots)))$ yields the same matrix.
  - Different multiplication orders do not cost the same:
    - Multiplying $p \times q$ matrix A and $q \times r$ matrix B takes $p \cdot q \cdot r$ multiplications; result is a $p \times r$ matrix.
    - Consider multiplying 10×100 matrix A_1 with 100×5 matrix A_2 and 5×50 matrix A_3.
      - $(A_1 \cdot A_2) \cdot A_3$ takes $10 \cdot 100 \cdot 5 + 10 \cdot 5 \cdot 50 = 7500$ multiplications.
      - $A_1 \cdot (A_2 \cdot A_3)$ takes $100 \cdot 5 \cdot 50 + 10 \cdot 50 \cdot 100 = 75000$ multiplications.
- In general, let A_i be $p_{i-1} \times p_i$ matrix.
  - $A_1, A_2, A_3, \ldots, A_n$ can be represented by $p_0, p_1, p_2, p_3, \ldots, p_n$
- Let $m(i, j)$ denote minimal number of multiplications needed to compute $A_i \cdot A_{i+1} \cdots A_j$
  - We want to compute $m(1, n)$.
- Divide-and-conquer solution/recursive algorithm:
  - Divide into $j - i - 1$ subproblems by trying to set parenthesis in all $j - i - 1$ positions. (e.g. $(A_i \cdot A_{i+1} \cdots A_k) \cdot (A_{k+1} \cdots A_j)$ corresponds to multiplying $p_{i-1} \times p_k$ and $p_k \times p_j$ matrices.)
  - Recursively find best way of solving sub-problems. (e.g. best way of computing $A_i \cdot A_{i+1} \cdots A_k$ and $A_{k+1} \cdot A_{k+2} \cdots A_j$)
  - Pick best solution.

- Algorithm expressed in terms of $m(i,j)$:

$$m(i,j) = \begin{cases} 0 & \text{If } i = j \\ \min_{i \le k < j}\{m(i,k) + m(k+1,j) + p_{i-1} \cdot p_k \cdot p_j\} & \text{If } i < j \end{cases}$$

- Program:

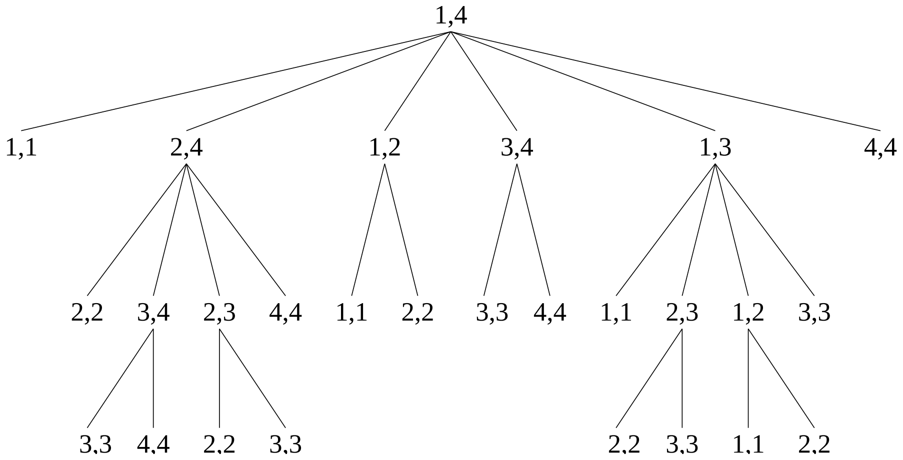
```
MATRIX-CHAIN($i,j$)
        IF $i = j$ THEN return 0
        $m(i,j) = \infty$
        FOR $k = i$ TO $j-1$ DO
            q = MATRIX-CHAIN($i,k$) + MATRIX-CHAIN($k+1,j$) +$p_{i-1} \cdot p_k \cdot p_j$
            IF $q < m(i,j)$ THEN $m(i,j) = q$
        OD
        Return $m(i,j)$
END MATRIX-CHAIN

Return MATRIX-CHAIN($1,n$)
```

- Running time:

$$\begin{aligned} T(n) &= \sum_{k=1}^{n-1}(T(k) + T(n-k) + O(1)) \\ &= 2 \cdot \sum_{k=1}^{n-1} T(k) + O(n) \\ &\ge 2 \cdot T(n-1) \\ &\ge 2 \cdot 2 \cdot T(n-2) \\ &\ge 2 \cdot 2 \cdot 2 \ldots \\ &= 2^n \end{aligned}$$

- Problem is that we compute the same result over and over again.
    - Example: Recursion tree for MATRIX-CHAIN$(1,4)$



We for example compute MATRIX-CHAIN$(3,4)$ twice

- Solution is to "remember" values we have already computed in a table—*memoization*

```
MATRIX-CHAIN($i, j$)
        IF $i = j$ THEN return 0
        IF $m(i, j) < \infty$ THEN return $m(i, j)$    /* This line has changed */
        FOR $k = i$ to $j - 1$ DO
                q = MATRIX-CHAIN($i, k$) + MATRIX-CHAIN($k + 1, j$)+$p_{i-1} \cdot p_k \cdot p_j$
                IF $q < m(i, j)$ THEN $m(i, j) = q$
        OD
        return $m(i, j)$
END MATRIX-CHAIN

FOR $i = 1$ to $n$ DO
        FOR $j = i$ to $n$ DO
                $m(i, j) = \infty$
        OD
OD

return MATRIX-CHAIN($1, n$)
```

- Running time:
  - $\Theta(n^2)$ different calls to MATRIX-CHAIN(i,j).
  - The first time a call is made it takes $O(n)$ time, *not* counting recursive calls.
  - When a call has been made once it costs $O(1)$ time to make it again.

    $\Downarrow$

    $O(n^3)$ time

  - Another way of thinking about it: $\Theta(n^2)$ total entries to fill, it takes $O(n)$ to fill one.

2 Alternative view of Dynamic Programming

- Often (including in the book) dynamic programming is presented in a different way; As filling up a table from the bottom.
- Matrix-chain example: Key is that $m(i,j)$ only depends on $m(i,k)$ and $m(k+1,j)$ where $i \leq k < j \Rightarrow$ if we have computed them, we can compute $m(i,j)$
  - We can easily compute $m(i,i)$ for all $1 \leq i \leq n$ $(m(i,i) = 0)$
  - Then we can easily compute $m(i,i+1)$ for all $1 \leq i \leq n-1$
    $m(i,i+1) = m(i,i) + m(i+1,i+1) + p_{i-1} \cdot p_i \cdot p_{i+1}$
  - Then we can compute $m(i,i+2)$ for all $1 \leq i \leq n-2$
    $m(i,i+2) = \min\{m(i,i) + m(i+1,i+2) + p_{i-1} \cdot p_i \cdot p_{i+2}, m(i,i+1) + m(i+2,i+2) + p_{i-1} \cdot p_{i+1} \cdot p_{i+2}\}$
    $\vdots$
  - Until we compute $m(1,n)$
  - Computation order:

j

i

	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2		1	2	3	4	5	6
3			1	2	3	4	5
4				1	2	3	4
5					1	2	3
6						1	2
7							1

− Computation order

- Program:

```
FOR $i = 1$ to $n$ DO
    $m(i,i) = 0$
OD
FOR $l = 1$ to $n-1$ DO
    FOR $i = 1$ to $n-l$ DO
        $j = i + l$
        $m(i,j) = \infty$
        FOR $k = 1$ to $j-1$ DO
            $q = m(i,k) + m(k+1,j) + p_{i-1} \cdot p_k \cdot p_j$
            IF $q < m(i,j)$ THEN $m(i,j) = q$
        OD
    OD
OD
```

- Analysis:
  - $O(n^2)$ entries, $O(n)$ time to compute each $\Rightarrow O(n^3)$.
- Note:
  - I like recursive (divide-and-conquer) thinking.
  - Book seems to like table method better.