Divide-and-conquer

Divide-and-Conquer (Input: Problem P)

To Solve P:

1. *Divide* P into smaller problems $P_1, P_2, P_3 .....P_k$.
2. *Conquer* by solving the (smaller) subproblems recursively.
3. *Combine* solutions to $P_1, P_2, ...P_k$ into solution for P.

1 MergeSort

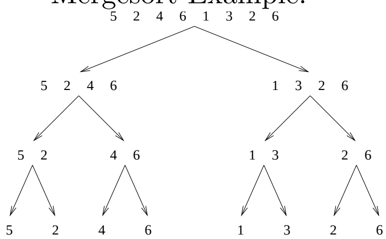
- Can we design better than n^2 (quadratic) sorting algorithm?
- We will do so using one of the most powerful algorithm design techniques.
- Using divide-and-conquer, we can obtain a mergesort algorithm.
  - Divide: Divide n elements into two subsequences of $n/2$ elements each.
  - Conquer: Sort the two subsequences recursively.
  - Combine: Merge the two sorted subsequences.
- Assume we have procedure Merge(A, p, q, r) which merges sorted A[p..q] with sorted A[q+1....r]
- We can sort A[p...r] as follows (initially p=1 and r=n):

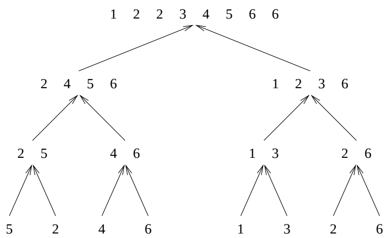
```
Merge Sort(A,p,r)
    If p < r then
        q = ⌊(p + r)/2⌋
        MergeSort(A,p,q)
        MergeSort(A,q+1,r)
        Merge(A,p,q,r)
```

- How does Merge(A, p, q, r) work?
  - Imagine merging two sorted piles of cards. The basic idea is to choose the smallest of the two top cards and put it into the output pile.
  - Running time: $(r - p)$

– Implementation is a bit messier..

Mergesort Example:





1.1 Mergesort Correctness

- Prove that Merge() is correct (what is the invariant?)
- Assuming that Merge is correct, prove that Mergesort() is correct.
  - Induction on n

1.2 Mergesort Analysis

- To simplify things, let us assume that n is a power of 2, i.e $n = 2^k$ for some k.
- Running time of a recursive algorithm can be analyzed using a **recurrence equation/relation**. Each "divide" step yields two sub-problems of size $n/2$.

$$\begin{aligned} T(n) &\leq c_1 + T(n/2) + T(n/2) + c_2 n \\ &\leq 2T(n/2) + (c_1 + c_2 n) \end{aligned}$$

- Next class we will prove that $T(n) \leq cn \log_2 n$. Intuitively, we can see why the recurrence has solution $n \log_2 n$ by looking at the **recursion tree**: the total number of levels in the recursion tree is $\log_2 n + 1$ and each level costs linear time.
- Note: If $n \neq 2^k$ the recurrence gets more complicated, but the solution is the same. (We will often assume $n = 2^k$ to avoid complicated cases).

2 Matrix Multiplication

- Let X and Y be $n \times n$ matrices

  $$X = \left\{ \begin{array}{cccc} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{1n} \\ x_{31} & x_{32} & \cdots & x_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{array} \right\}$$

- We want to compute $Z = X \cdot Y$
  - $z_{ij} = \sum_{k=1}^{n} X_{ik} \cdot Y_{kj}$
- Naive method uses $\Rightarrow n^2 \cdot n = \Theta(n^3)$ operations
- Divide-and-conquer solution:

  $$Z = \left\{ \begin{array}{cc} A & B \\ C & D \end{array} \right\} \cdot \left\{ \begin{array}{cc} E & F \\ G & H \end{array} \right\} = \left\{ \begin{array}{cc} (A \cdot E + B \cdot G) & (A \cdot F + B \cdot H) \\ (C \cdot E + D \cdot G) & (C \cdot F + D \cdot H) \end{array} \right\}$$

  - The above naturally leads to divide-and-conquer solution:
    - Divide X and Y into 8 sub-matrices A, B, C, and D.
    - Do 8 matrix multiplications recursively.
    - Compute Z by combining results (doing 4 matrix additions).
  - Lets assume $n = 2^c$ for some constant c and let A, B, C and D be $n/2 \times n/2$ matrices
    - Running time of algorithm is $T(n) = 8T(n/2) + \Theta(n^2) \Rightarrow T(n) = \Theta(n^3)$
  - But we already discussed a (simpler/naive) $O(n^3)$ algorithm! Can we do better?

2.1 Strassen's Algorithm

- Strassen observed the following:

  $$Z = \left\{ \begin{array}{cc} A & B \\ C & D \end{array} \right\} \cdot \left\{ \begin{array}{cc} E & F \\ G & H \end{array} \right\} = \left\{ \begin{array}{cc} (S_1 + S_2 - S_4 + S_6) & (S_4 + S_5) \\ (S_6 + S_7) & (S_2 + S_3 + S_5 - S_7) \end{array} \right\}$$

  where

$$\begin{array}{rcl} S_1 & = & (B - D) \cdot (G + H) \\ S_2 & = & (A + D) \cdot (E + H) \\ S_3 & = & (A - C) \cdot (E + F) \\ S_4 & = & (A + B) \cdot H \\ S_5 & = & A \cdot (F - H) \\ S_6 & = & D \cdot (G - E) \\ S_7 & = & (C + D) \cdot E \end{array}$$

– Lets test that $S_6 + S_7$ is really $C \cdot E + D \cdot G$

$$\begin{aligned} S_6 + S_7 &= D \cdot (G - E) + (C + D) \cdot E \\ &= DG - DE + CE + DE \\ &= DG + CE \end{aligned}$$

- This leads to a divide-and-conquer algorithm with running time $T(n) = 7T(n/2) + \Theta(n^2)$
  – We only need to perform 7 multiplications recursively.
  – Division/Combination can still be performed in $\Theta(n^2)$ time.
- Lets solve the recurrence using the iteration method

$$\begin{aligned} T(n) &= 7T(n/2) + n^2 \\ &= n^2 + 7(7T(\frac{n}{2^2}) + (\frac{n}{2})^2) \\ &= n^2 + (\frac{7}{2^2})n^2 + 7^2T(\frac{n}{2^2}) \\ &= n^2 + (\frac{7}{2^2})n^2 + 7^2(7T(\frac{n}{2^3}) + (\frac{n}{2^2})^2) \\ &= n^2 + (\frac{7}{2^2})n^2 + (\frac{7}{2^2})^2 \cdot n^2 + 7^3T(\frac{n}{2^3}) \\ &= n^2 + (\frac{7}{2^2})n^2 + (\frac{7}{2^2})^2 n^2 + (\frac{7}{2^2})^3 n^2 .... + (\frac{7}{2^2})^{\log n - 1} n^2 + 7^{\log n} \\ &= \sum_{i=0}^{\log n - 1} (\frac{7}{2^2})^i n^2 + 7^{\log n} \\ &= n^2 \cdot \Theta((\frac{7}{2^2})^{\log n - 1}) + 7^{\log n} \\ &= n^2 \cdot \Theta(\frac{7^{\log n}}{(2^2)^{\log n}}) + 7^{\log n} \\ &= n^2 \cdot \Theta(\frac{7^{\log n}}{n^2}) + 7^{\log n} \\ &= \Theta(7^{\log n}) \end{aligned}$$

  – Now we have the following:

$$\begin{aligned} 7^{\log n} &= 7^{\frac{\log_7 n}{\log_7 2}} \\ &= (7^{\log_7 n})^{(1/\log_7 2)} \\ &= n^{(1/\log_7 2)} \\ &= n^{\frac{\log_2 7}{\log_2 2}} \\ &= n^{\log 7} \end{aligned}$$

  – Or in general: $a^{\log_k n} = n^{\log_k a}$

So the solution is $T(n) = \Theta(n^{\log 7}) = \Theta(n^{2.81\ldots})$

- Note:
  - We are 'hiding' a much bigger constant in $\Theta()$ than before.
  - Currently best known bound is $O(n^{2.376..})$ (another method).
  - Lower bound is (trivially) $\Omega(n^2)$.