# Splay Trees

Handout

## 1 Amortized Analysis

- Last time we discussed amortized analysis of data structures
  - A way of expressing that even though the worst-case performance of an operation can be bad, the total performance of a sequence of operations cannot be too bad.
- One way of thinking of amortized time is as being an "average": If any sequence of $n$ operations takes less than $T(n)$ time, the amortized time per operation is $T(n)/n$.
- We formally defined amortized time using the idea that we over-charge some operations and store the over-charge as credits/potential that can then help pay for later operations (*potential method*)
  - Consider performing $n$ operations on an initial data structure $D_0$
  - $D_i$ is data structure after $i$th operation.
  - $c_i$ is actual cost (time) of $i$th operation.
  - Potential function: $\Phi : D_i \rightarrow R$
  - $\tilde{c}_i$ amortized cost of $i$th operation: $\tilde{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$
  - Given $\Phi(D_0) = 0$ and $\Phi(D_i) \geq 0$: $\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \tilde{c}_i$
- We also discussed two examples of amortized analysis
  - Stack with Multipop ($O(n)$ worst-case, $O(1)$ amortized).
  - Increment on binary counter ($O(\log n)$ worst-case, $O(1)$ amortized).

  In both cases we could argue for $O(1)$ amortized performance without actually doing potential calculation—we just think about potential/credits as being distributed on certain parts of the data structure and let operations put and take credits while maintaining some invariant (*accounting method*).

## 2 Splay trees

- We have previously discussed binary search trees and how they can be kept balanced ($O(\log n)$ height) during insert and delete operations (red-black trees).
  - Rebalancing rather complicated
  - Extra space used for the color of each node
- We also discussed skip lists which are a lot simpler than red-black trees
  - Only guarantee $O(\log n)$ *expected* performance
  - No extra information is used for rebalance information though
- Splay trees are search trees that "magically" balance themselves (no rebalance information is stored) and have *amortized* $O(\log n)$ performance.
- Recall search trees:
  - Binary tree with elements in nodes
  - If node $v$ holds element $e$ then
    - all elements in left subtree $< e$
    - all elements in left subtree $> e$
- Splay tree:
  - Normal (possibly unbalanced) search tree $T$
  - All operations implemented using one basic operation, Splay:

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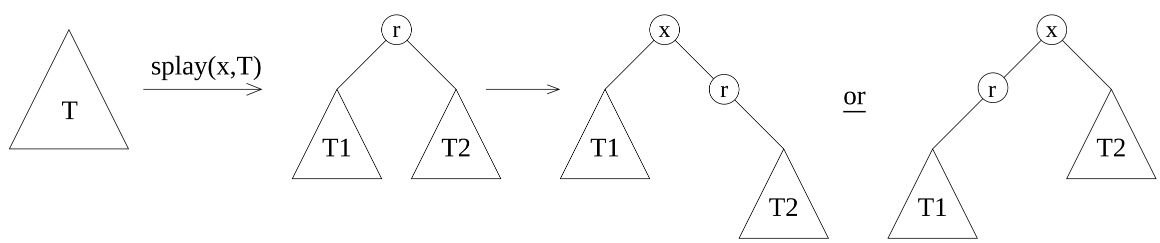
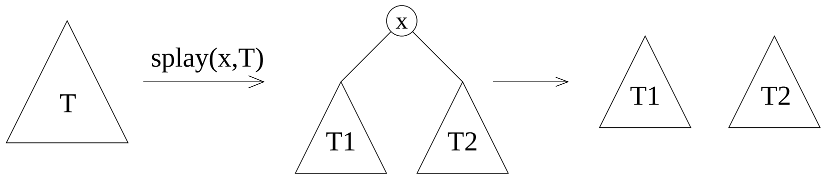
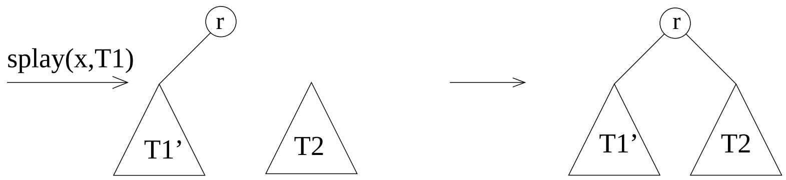
| Splay$(x, T)$ searches for $x$ in $T$ and reorganizes tree such that $x$ (or min element $> x$ or max element $< x$) is in root |

  - Search$(x, T)$: Splay$(x, T)$ and inspect root
  - Insert$(x, T)$: Splay$(x, T)$ and create new root with $x$



- DELETE$(x, T)$:
  - SPLAY$(x, T)$ and remove root $\rightarrow$ tree falls into $T1$ and $T2$.
  - SPLAY$(x, T1)$
  - Make $T2$ right son of new root of $T1$ after splay

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All operations perform $O(1)$ SPLAY's and use $O(1)$ extra time.

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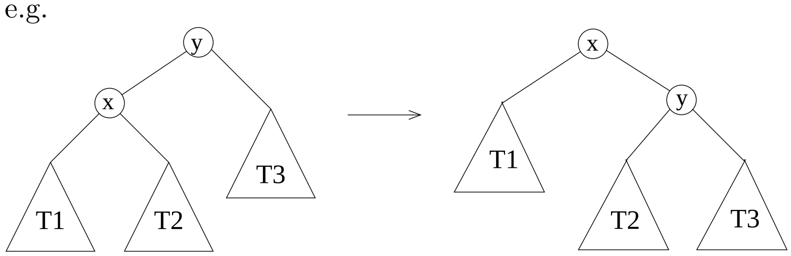
$O(\log n)$ amortized SPLAY gives $O(\log n)$ amortized bound on all operations.

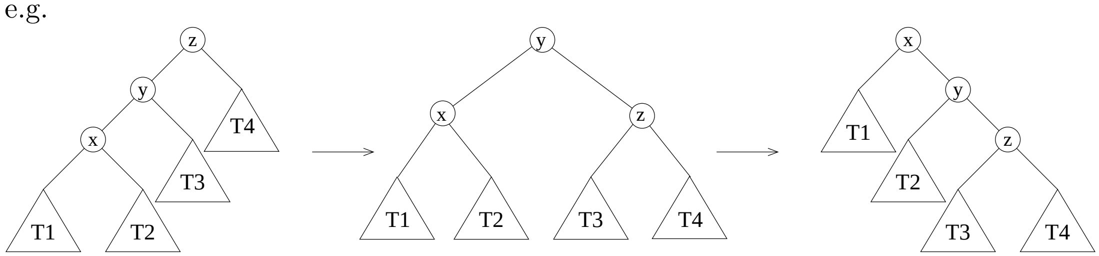
- Implementation of SPLAY:
  - Search for $x$ like in normal search tree
  - Repeatedly rotate $x$ up until it becomes the root.
    We distinguish between three cases:
    1. $x$ is child of root (no grandparent): **rotate($x$)**

       e.g.



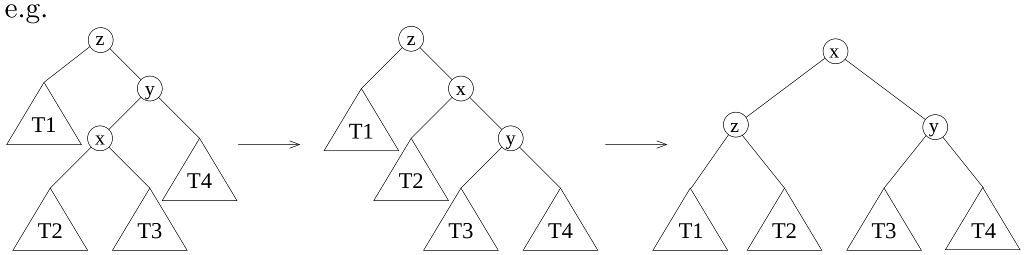

    2. $x$ has parent $y$ *and* grandparent $z$ *and* both $x$ and $y$ left (right) children: **rotate($y$) followed by rotate($x$)** Note: Does not work with rotate(x) and rotate(x)

       e.g.



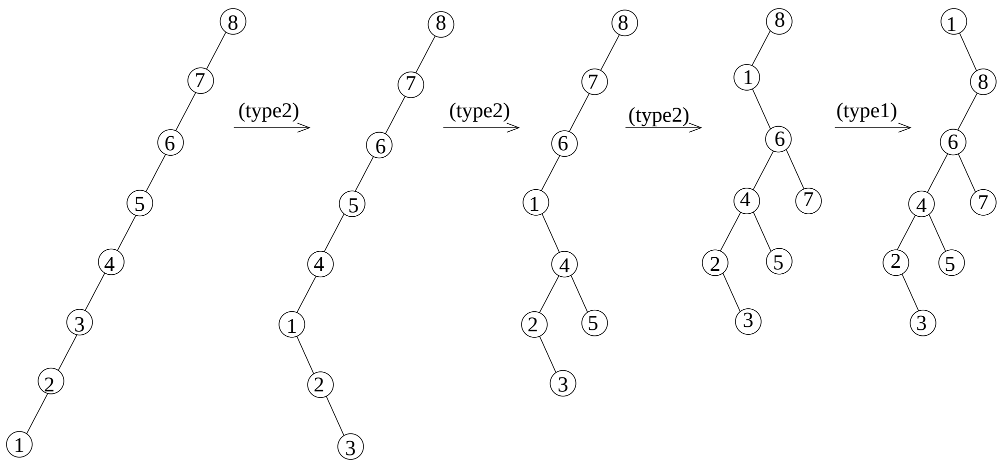
3. $x$ has parent $y$ *and* grandparent $z$ *and* one of $x$ and $y$ is a left child and the other is a right child: **rotate($x$) followed by rotate($x$)**

e.g.

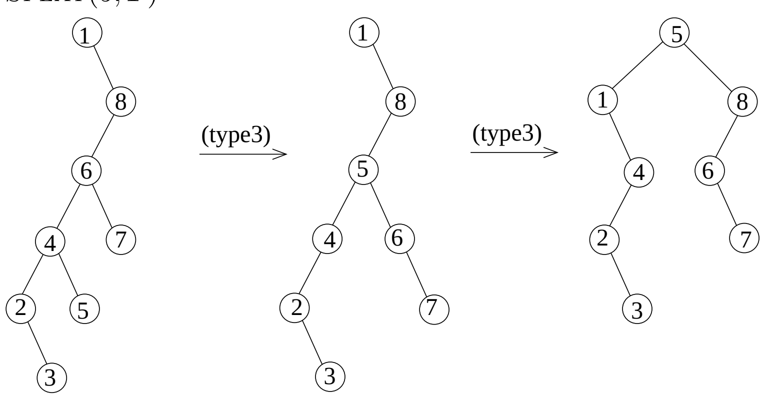



- Note:
  - A SPLAY can take $O(n)$ worst-case time (very unbalanced tree)
  - But Splay trees somehow seem to stay nicely balanced

  Examples: SPLAY$(1, T)$




SPLAY$(5, T)$



- Analysis:
  - We will use *accounting method* to show that all operations (SPLAY) takes $O(\log n)$ amortized time.
    - We will imagine that each node in tree has credits on it
    - We will use some credits to pay for (part of) rotations during a splay
    - We will see that we only have to place $O(\log n)$ new credits (on root) when performing an INSERT or DELETE
  - Note that we will ignore cost of searching for $x$, since the rotations cost at least as much as the search ($\Rightarrow$ if we can bound amortized rotation cost we also bound search cost).
  - Let $T(x)$ be tree rooted at $x$. We will maintain the *credit invariant* that each node $x$ holds $\mu(x) = \lfloor \log |T(x)| \rfloor$ credits.
  - We will prove the following lemma:

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    |---|
    | Less than or equal to $3(\mu(T)-\mu(x)+O(1))$ credits are needed to perform SPLAY$(x, T)$ operation and maintain credit invariant |

  - Using this lemma we get that a SPLAY operation uses at most $3\lfloor \log n \rfloor + O(1) = O(\log n)$ credits (time).
  - As an INSERT or a DELETE requires us to insert at most $O(\log n)$ extra credits (on the root) more than the ones used on the SPLAY, we get the $O(\log n)$ amortized bound.
- Proof of lemma:
  - Let $\mu$ and $\mu'$ be the value of $\mu$ before and after a rotate operation in case 1, 2, or 3.
  - During a SPLAY operation we perform a number of, say $k \geq 0$, case 2 and 3 operations and possibly a case 1 operation.
  - Next time we will show that the cost of one operation is:
    - Case 1: $3(\mu'(x) - \mu(x) + O(1))$
    - Case 2: $3(\mu'(x) - \mu(x))$
    - Case 3: $3(\mu'(x) - \mu(x))$

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    When we sum over all $\leq k+1$ operations in a splay we get $3(\mu(T) - \mu(x) + O(1))$ where $\mu(x)$ is the number of credits on $x$ before the SPLAY.

    Note that it is important that we only have the $O(1)$ term in case 1.
- Case 1:
  - We have: $\mu'(x) = \mu(y)$, $\mu'(y) \leq \mu'(x)$ and all other $\mu$'s are unchanged.
  - To maintain invariant we use:
    $$\begin{aligned} \mu'(x) + \mu'(y) - \mu(x) - \mu(y) &= \mu'(y) - \mu(x) \\ &\leq \mu'(x) - \mu(x) \\ &\leq 3(\mu'(x) - \mu(x)) \end{aligned}$$
  - To do actual rotation we use $O(1)$ credits.

- Case 2:
  - We have $\mu'(x) = \mu(z)$, $\mu'(y) \leq \mu'(x)$, $\mu'(z) \leq \mu'(x)$, $\mu(y) \geq \mu(x)$ and all other $\mu$'s are unchanged.
  - To maintain invariant we use:

$$\begin{aligned} \mu'(x) + \mu'(y) + \mu'(z) - \mu(x) - \mu(y) - \mu(z) &= \mu'(y) + \mu'(z) - \mu(x) - \mu(y) \\ &= (\mu'(y) - \mu(x)) + (\mu'(z) - \mu(y)) \\ &\leq (\mu'(x) - \mu(x)) + (\mu'(x) - \mu(x)) \\ &= 2(\mu'(x) - \mu(x)) \end{aligned}$$

  - This means that we can use the remaining $\mu'(x) - \mu(x)$ credits to pay for rotation, *unless* $\mu'(x) = \mu(x)$ (can happen since $\mu(x) = \lfloor \log |T(x)| \rfloor$).
  - We will show that if $\mu'(x) = \mu(x)$ then $\mu'(x) + \mu'(y) + \mu'(z) < \mu(x) + \mu(y) + \mu(z)$ which means that the operation actually *releases* credits we can use for the rotation:
    * Assume $\mu'(x) = \mu(x)$ and $\mu'(x) + \mu'(y) + \mu'(z) \geq \mu(x) + \mu(y) + \mu(z)$
    * We have $\mu(z) = \mu'(x) = \mu(x)$

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      $\mu(z) = \mu(x) = \mu(y)$

      and $\mu'(x) + \mu'(y) + \mu'(z) \geq \mu(x) + \mu(y) + \mu(z) = 3\mu(x) = 3\mu'(x)$

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      $\mu'(y) + \mu'(z) \geq 2\mu'(x)$
    * Since $\mu'(y) \leq \mu'(x)$ and $\mu'(z) \leq \mu'(x)$ we get $\mu'(x) = \mu'(y) = \mu'(z)$
    * Since $\mu(z) = \mu'(x)$ we have $\mu(x) = \mu(y) = \mu(z) = \mu'(x) = \mu'(y) = \mu'(z)$ which cannot be true (and thus our initial assumption cannot be true):
      Let $a$ be $|T(x)|$ before rotations ($a = |T1| + |T2| + 1$)
      Let $b$ be $|T(z)|$ after rotations ($b = |T3| + |T4| + 1$)
      Since $\mu(x) = \mu'(z) = \mu'(x)$ we have $\lfloor \log a \rfloor = \lfloor \log b \rfloor = \lfloor \log(a + b + 1) \rfloor$ but then we have the following contradiction:
      · if $a \leq b$: $\lfloor \log(a + b + 1) \rfloor \geq \lfloor \log 2a \rfloor = 1 + \lfloor \log a \rfloor > \lfloor \log a \rfloor$
      · if $a > b$: $\lfloor \log(a + b + 1) \rfloor \geq \lfloor \log 2b \rfloor = 1 + \lfloor \log b \rfloor > \lfloor \log b \rfloor$

- Case 3:
  - Can be proved analogously to case 2.