Chapter 2
Basics of Algorithm Analysis

2.1 Computational Tractability

As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise - By what course of calculation can these results be arrived at by the machine in the shortest time? - Charles Babbage

"For me, great algorithms are the poetry of computation. Just like verse, they can be terse, allusive, dense, and even mysterious. But once unlocked, they cast a brilliant new light on some aspect of computing." - Francis Sullivan

Computational Tractability

Polynomial-Time

Brute force. For many non-trivial problems, there is a natural brute force search algorithm that checks every possible solution.
- Typically takes $2^n$ time or worse for inputs of size $N$.
- Unacceptable in practice.

Desirable scaling property. When the input size doubles, the algorithm should only slow down by some constant factor $C$.

There exists constants $c > 0$ and $d > 0$ such that on every input of size $N$, its running time is bounded by $cN^d$ steps.

Def. An algorithm is poly-time if the above scaling property holds.
**Worst-Case Analysis**

**Worst case running time.** Obtain bound on largest possible running time of algorithm on input of a given size $N$.
- Generally captures efficiency in practice.
- Draconian view, but hard to find effective alternative.

**Average case running time.** Obtain bound on running time of algorithm on random input as a function of input size $N$.
- Hard (or impossible) to accurately model real instances by random distributions.
- Algorithm tuned for a certain distribution may perform poorly on other inputs.

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**Worst-Case Polynomial-Time**

**Def.** An algorithm is **efficient** if its running time is polynomial.

**Justification:** It really works in practice!
- Although $6.02 \times 10^{23} \times N^{20}$ is technically poly-time, it would be useless in practice.
- In practice, the poly-time algorithms that people develop almost always have low constants and low exponents.
- Breaking through the exponential barrier of brute force typically exposes some crucial structure of the problem.

**Exceptions.**
- Some poly-time algorithms do have high constants and/or exponents, and are useless in practice.
- Some exponential-time (or worse) algorithms are widely used because the worst-case instances seem to be rare.

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**Why It Matters**

### 2.2 Asymptotic Order of Growth

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n$</th>
<th>$n^{\log n}$</th>
<th>$n^2$</th>
<th>$n^3$</th>
<th>$1.5^n$</th>
<th>$2^n$</th>
<th>$n!$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 10$</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>4 sec</td>
<td></td>
</tr>
<tr>
<td>$n = 30$</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>18 min</td>
<td>$10^{15}$ years</td>
<td></td>
</tr>
<tr>
<td>$n = 50$</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>11 min</td>
<td>36 years</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 100$</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>1 sec</td>
<td>1,289 years</td>
<td>$10^{17}$ years</td>
</tr>
<tr>
<td>$n = 1,000$</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>1 sec</td>
<td>18 min</td>
<td>very long</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 10,000$</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>2 min</td>
<td>12 days</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 100,000$</td>
<td>&lt; 1 sec</td>
<td>2 sec</td>
<td>3 hours</td>
<td>32 years</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 1,000,000$</td>
<td>1 sec</td>
<td>20 sec</td>
<td>12 days</td>
<td>31,710 years</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
</tbody>
</table>

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds $10^{23}$ years, we simply record the algorithm as taking a very long time.
Asymptotic Order of Growth

**Upper bounds.** \( T(n) = O(f(n)) \) if there exist constants \( c > 0 \) and \( n_0 \) such that for all \( n \geq n_0 \) we have \( T(n) \leq c \cdot f(n) \).

**Lower bounds.** \( T(n) = \Omega(f(n)) \) if there exist constants \( c > 0 \) and \( n_0 \) such that for all \( n \geq n_0 \) we have \( T(n) \geq c \cdot f(n) \).

**Tight bounds.** \( T(n) = \Theta(f(n)) \) if \( T(n) \) is both \( O(f(n)) \) and \( \Omega(f(n)) \).

Ex: \( T(n) = 32n^2 + 17n + 32 \).

- \( T(n) \) is \( O(n^2) \), \( O(n^3) \), \( \Omega(n^2) \), \( \Theta(n^2) \).
- \( T(n) \) is not \( O(n) \), \( \Omega(n^3) \), \( \Theta(n) \), or \( \Theta(n^3) \).

Slight abuse of notation. \( T(n) = O(f(n)) \).

- Not transitive:
  - \( f(n) = 5n^3 \); \( g(n) = 3n^2 \)
  - \( f(n) = O(n^3) \); \( g(n) \)
  - but \( f(n) = g(n) \).

- Better notation: \( T(n) \in O(f(n)) \).

Meaningless statement. Any comparison-based sorting algorithm requires at least \( O(n \log n) \) comparisons.

- Statement doesn’t “type-check.”
- Use \( \Omega \) for lower bounds.

Properties

**Transitivity.**
- If \( f = O(g) \) and \( g = O(h) \) then \( f = O(h) \).
- If \( f = \Omega(g) \) and \( g = \Omega(h) \) then \( f = \Omega(h) \).
- If \( f = \Theta(g) \) and \( g = \Theta(h) \) then \( f = \Theta(h) \).

**Additivity.**
- If \( f = O(g) \) and \( g = O(h) \) then \( f + g = O(h) \).
- If \( f = \Omega(g) \) and \( g = \Omega(h) \) then \( f + g = \Omega(h) \).
- If \( f = \Theta(g) \) and \( g = \Omega(h) \) then \( f + g = \Theta(h) \).

Asymptotic Bounds for Some Common Functions

**Polynomials.** \( a_0 + a_1n + \ldots + a_dn^d \) is \( \Theta(n^d) \) if \( a_d \) is positive.

**Polynomial time.** Running time is \( O(n^d) \) for some constant \( d \) independent of the input size \( n \).

**Logarithms.** \( O(\log_a n) = O(\log_b n) \) for any constants \( a, b \) with \( a, b > 0 \).

**Logarithms.** For every \( x > 0 \), \( \log n = O(n^\epsilon) \).

- \( \log \) grows slower than every polynomial

**Exponentials.** For every \( r > 1 \) and every \( d > 0 \), \( n^d = O(r^n) \).

- Every exponential grows faster than every polynomial
2.4 A Survey of Common Running Times

Linear Time: \( O(n) \)

Linear time. Running time is proportional to input size.

Computing the maximum. Compute maximum of \( n \) numbers \( a_1, \ldots, a_n \).

\[
\begin{align*}
\text{max } & \leftarrow a_1 \\
\text{for } i = 2 \text{ to } n \{ \\
\quad \text{if } (a_i > \text{max}) \\
\quad \quad \text{max } \leftarrow a_i \\
\}\}
\]

Merge. Combine two sorted lists \( A = a_1, a_2, \ldots, a_n \) with \( B = b_1, b_2, \ldots, b_n \) into sorted whole.

\[
\begin{align*}
\text{Merged result} & \\
\end{align*}
\]

\[
\begin{align*}
i & = 1, \ j = 1 \\
\text{while } (\text{both lists are nonempty}) \{ \\
\quad \text{if } (a_i \leq b_j) \text{ append } a_i \text{ to output list and increment } i \\
\quad \quad \text{else append } b_j \text{ to output list and increment } j \\
\}\} \\
\text{append remainder of nonempty list to output list}
\]

Claim. Merging two lists of size \( n \) takes \( O(n) \) time.

Pf. After each comparison, the length of output list increases by 1.

O(n log n) Time

O(n log n) time. Arises in divide-and-conquer algorithms.

also referred to as linearithmic time

Sorting. Mergesort and heapsort are sorting algorithms that perform \( O(n \log n) \) comparisons.

Largest empty interval. Given \( n \) time-stamps \( x_1, \ldots, x_n \) on which copies of a file arrive at a server, what is largest interval of time when no copies of the file arrive?

\( O(n \log n) \) solution. Sort the time-stamps. Scan the sorted list in order, identifying the maximum gap between successive time-stamps.
Polynomial Time: $O(nk)$ Time

Independent set of size $k$. Given a graph, are there $k$ nodes such that no two are joined by an edge?

$O(n^k)$ solution. Enumerate all subsets of $k$ nodes.

\[
\text{foreach subset } S \text{ of } k \text{ nodes} \\
\quad \text{check whether } S \text{ in an independent set} \\
\quad \text{if } (S \text{ is an independent set}) \\
\quad \quad \text{report } S \text{ is an independent set} \\
\}
\]

• Check whether $S$ is an independent set = $O(k^2)$.
• Number of $k$ element subsets = \[ n \begin{pmatrix} n-k+1 \\ k \end{pmatrix} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k(k-1)(k-2)\ldots(2)(1)} = \frac{n^k}{k!} \] poly-time for $k=17$, but not practical

Quadratic Time: $O(n^2)$

Quadratic time. Enumerate all pairs of elements.

Closest pair of points. Given a list of $n$ points in the plane $(x_1, y_1), \ldots, (x_n, y_n)$, find the pair that is closest.

$O(n^2)$ solution. Try all pairs of points.

\[
\min \left\{ (x_i - x_j)^2 + (y_i - y_j)^2 \right\} \\
\text{for } i = 1 \text{ to } n \{ \\
\quad \text{for } j = i+1 \text{ to } n \{ \\
\quad \quad d \leftarrow (x_i - x_j)^2 + (y_i - y_j)^2 \\
\quad \quad \text{if } (d < \min) \\
\quad \quad \quad \min \leftarrow d \\
\quad \}\}
\]

Remark. $\Omega(n^2)$ seems inevitable, but this is just an illusion. see chapter 5

Cubic Time: $O(n^3)$

Cubic time. Enumerate all triples of elements.

Set disjointness. Given $n$ sets $S_1, \ldots, S_n$ each of which is a subset of $1, 2, \ldots, n$, is there some pair of these which are disjoint?

$O(n^3)$ solution. For each pairs of sets, determine if they are disjoint.

\[
\text{foreach set } S_i \{ \\
\quad \text{foreach other set } S_j \{ \\
\quad \quad \text{foreach element } p \text{ of } S_i \{ \\
\quad \quad \quad \text{determine whether } p \text{ also belongs to } S_j \\
\quad \quad \quad \text{if } (\text{no element of } S_i \text{ belongs to } S_j) \\
\quad \quad \quad \quad \text{report that } S_i \text{ and } S_j \text{ are disjoint} \\
\quad \}\}
\}
\]

Exponential Time

Independent set. Given a graph, what is maximum size of an independent set?

$O(n^2 2^n)$ solution. Enumerate all subsets.

\[
S^* \leftarrow \emptyset \\
\text{foreach subset } S \text{ of nodes} \{ \\
\quad \text{check whether } S \text{ in an independent set} \\
\quad \text{if } (S \text{ is largest independent set seen so far}) \\
\quad \quad \text{update } S^* \leftarrow S \\
\}
\]