Quicksort
(CLRS 7)

• We previously saw how the divide-and-conquer technique can be used to design sorting algorithm—Merge-sort
  – Partition $n$ elements array $A$ into two subarrays of $n/2$ elements each
  – Sort the two subarrays recursively
  – Merge the two subarrays
Running time: $T(n) = 2T(n/2) + \Theta(n) \Rightarrow T(n) = \Theta(n \log n)$

• Another possibility is to divide the elements such that there is no need of merging, that is
  – Partition $A[1...n]$ into subarrays $A' = A[1..q]$ and $A'' = A[q+1...n]$ such that all elements in $A''$ are larger than all elements in $A'$.
  – Recursively sort $A'$ and $A''$.
  – (nothing to combine/merge. $A$ already sorted after sorting $A'$ and $A''$)

• Pseudo code for Quicksort:

```
QUICKSORT(A, p, r)
IF p < r THEN
    q = PARTITION(A, p, r)
    QUICKSORT(A, p, q - 1)
    QUICKSORT(A, q + 1, r)
FI
```

Sort using QUICKSORT($A, 1, n$)

If $q = n/2$ and we divide in $\Theta(n)$ time, we again get the recurrence $T(n) = 2T(n/2) + \Theta(n)$ for the running time $\Rightarrow T(n) = \Theta(n \log n)$

The problem is that it is hard to develop partition algorithm which always divide $A$ in two halves
QuickSort correctness:

- Easy to show, inductively, if Partition works correctly

- Example:

  \[
  \begin{array}{|c|c|c|c|c|c|c|}
  \hline
  2 & 8 & 7 & 1 & 3 & 5 & 6 & 4 \\
  \hline
  2 & 8 & 7 & 1 & 3 & 5 & 6 & 4 \\
  2 & 8 & 7 & 1 & 3 & 5 & 6 & 4 \\
  2 & 8 & 7 & 1 & 3 & 5 & 6 & 4 \\
  2 & 8 & 7 & 1 & 3 & 5 & 6 & 4 \\
  \hline
  \end{array} \quad \begin{array}{c}
  i=0, j=1 \\
  i=1, j=2 \\
  i=1, j=3 \\
  i=1, j=4 \\
  i=2, j=5 \\
  i=3, j=6 \\
  i=3, j=7 \\
  i=3, j=8 \\
  q=4 \\
  \end{array}
  \]

- Partition can be proved correct (by induction) using the loop invariant:
  - \( A[k] \leq x \) for \( p \leq k \leq i \)
  - \( A[k] > x \) for \( i+1 \leq k \leq j-1 \)
  - \( A[k] = x \) for \( k = r \)

QuickSort analysis:

- Partition runs in time \( \Theta(r - p) \)
- Running time depends on how well Partition divides \( A \).
- In the example it does reasonably well.
- If array is always partitioned nicely in two halves (partition returns \( q = \frac{r-p}{2} \)), we have the recurrence \( T(n) = 2T(n/2) + \Theta(n) \Rightarrow T(n) = \Theta(n \log n) \).
- But, in the worst case Partition always returns \( q = p \) or \( q = r \) and the running time becomes \( T(n) = \Theta(n) + T(0) + T(n-1) \Rightarrow T(n) = \Theta(n^2) \).
and what is maybe even worse, the worst case is when A is already sorted.

- So why is it called "quick"-sort? Because it "often" performs very well—can we theoretically justify this?

- Even if all the splits are relatively bad, we get $\Theta(n \log n)$ time:
  * Example: Split is $\frac{9}{10}n$, $\frac{1}{10}n$.
    
    $T(n) = T(\frac{9}{10}n) + T(\frac{1}{10}n) + n$
    
    Solution?
    
    Guess: $T(n) \leq cn \log n$
    
    Induction
    
    $T(n) = T(\frac{9}{10}n) + T(\frac{1}{10}n) + n$
    
    $\leq \frac{9cn}{10} \log(\frac{9n}{10}) + \frac{cn}{10} \log(\frac{n}{10}) + n$
    
    $\leq \frac{9cn}{10} \log n + \frac{9cn}{10} \log(\frac{9}{10}) + \frac{cn}{10} \log n + \frac{cn}{10} \log(\frac{1}{10}) + n$
    
    $\leq \frac{9cn}{10} \log n - c\log 10 - \frac{9c}{10} \log 9 - 1$
    
    
    $T(n) \leq cn \log n$ if $c \log 10 - \frac{9c}{10} \log 9 - 1 > 0$ which is definitely true if $c > \frac{10}{\log 10}$
    
    - So, in other words, if the splits happen at a constant fraction of $n$ we get $\Theta(n \log n)$—or, it's almost never bad!

**Average running time**

The natural question is: what is the average case running time of QUICKSORT? Is it close to worst-case ($\Theta(n^2)$), or to the best case $\Theta(n \log n)$? Average time depends on the distribution of inputs for which we take the average.

- If we run QUICKSORT on a set of inputs that are all almost sorted, the average running time will be close to the worst-case.

- Similarly, if we run QUICKSORT on a set of inputs that give good splits, the average running time will be close to the best-case.

- If we run QUICKSORT on a set of inputs which are picked uniformly at random from the space of all possible input permutations, then the average case will also be close to the best-case. Why? Intuitively, if any input ordering is equally likely, then we expect at least as many good splits as bad splits, therefore on the average a bad split will be followed by a good split, and it gets “absorbed” in the good split.

So, under the assumption that all input permutations are equally likely, the average time of QUICKSORT is $\Theta(n \log n)$ (intuitively). Is this assumption realistic?
• Not really. In many cases the input is almost sorted; think of rebuilding indexes in a database etc.

The question is: how can we make QUICKSORT have a good average time irrespective of the input distribution?

• Using randomization.

Randomization

We consider what we call randomized algorithms, that is, algorithms that make some random choices during their execution.

• Running time of normal deterministic algorithm only depend on the input.
• Running time of a randomized algorithm depends not only on input but also on the random choices made by the algorithm.
• Running time of a randomized algorithm is not fixed for a given input!
• Randomized algorithms have best-case and worst-case running times, but the inputs for which these are achieved are not known, they can be any of the inputs.

We are normally interested in analyzing the expected running time of a randomized algorithm, that is, the expected (average) running time for all inputs of size $n$

$$T_e(n) = E_{|X|=n}[T(X)]$$

Randomized Quicksort

• We can enforce that all $n!$ permutations are equally likely by randomly permuting the input before the algorithm.

  – Most computers have pseudo-random number generator $\text{random}(1, n)$ returning “random” number between 1 and $n$
  – Using pseudo-random number generator we can generate a random permutation (such that all $n!$ permutations equally likely) in $O(n)$ time:
    Note: Just choosing $A[i]$ randomly among elements $A[1..n]$ for all $i$ will not give random permutation! Why?
Alternatively we can modify PARTITION slightly and exchange last element in $A$ with random element in $A$ before partitioning.

\[
\text{\textsc{RandPartition}}(A, p, r) \\
i = \text{\textsc{Random}}(p, r) \\
\text{Exchange } A[r] \text{ and } A[i] \\
\text{RETURN } \text{\textsc{Partition}}(A, p, r)
\]

\[
\text{\textsc{RandQuicksort}}(A, p, r) \\
\text{IF } p < r \text{ THEN} \\
\quad q = \text{\textsc{RandPartition}}(A, p, r) \\
\quad \text{\textsc{RandQuicksort}}(A, p, q - 1) \\
\quad \text{\textsc{RandQuicksort}}(A, q + 1, r) \\
\text{FI}
\]

**Expected Running Time of Randomized Quicksort**

Let $T(n)$ be the running time of \textsc{RandQuicksort} for an input of size $n$.

- Running time of \textsc{RandQuicksort} is the total running time spent in all \textsc{Partition} calls.
- \textsc{Partition} is called $n$ times
  - The pivot element $x$ is not included in any recursive calls.
- One call of \textsc{Partition} takes $O(1)$ time plus time proportional to the number of iterations of FOR-loop.
  - In each iteration of FOR-loop we compare an element with the pivot element.

\[\Downarrow\]

If $X$ is the number of comparisons $A[j] \leq x$ performed in \textsc{Partition} over the entire execution of \textsc{RandQuicksort} then the running time is $O(n + X)$.

\[\Downarrow\]

\[E[T(n)] = E[O(n + X)] = n + E[X]\]

\[\Downarrow\]

To analyze the expected running time we need to compute $E[X]$

- To compute $X$ we use $z_1, z_2, \ldots, z_n$ to denote the elements in $A$ where $z_i$ is the $i$th smallest element. We also use $Z_{ij}$ to denote \{\(z_i, z_{i+1}, \ldots, z_j\}\).
- Each pair of elements $z_i$ and $z_j$ are compared at most once (when either of them is the pivot)

\[\Downarrow\]

\[X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\] where
\[ X_{ij} = \begin{cases} 1 & \text{If } z_i \text{ compared to } z_j \\ 0 & \text{If } z_i \text{ not compared to } z_j \end{cases} \]

\[ E[X] = E \left[ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \right] \]

\[ = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}] \]

\[ = \sum_{i=1}^{n} \sum_{j=i+1}^{n} P_r[z_i \text{ compared to } z_j] \]

- To compute \( P_r[z_i \text{ compared to } z_j] \) it is useful to consider when two elements are not compared.

Example: Consider an input consisting of numbers 1 through \( n \). Assume first pivot it 7 ⇒ first partition separates the numbers into sets \{1, 2, 3, 4, 5, 6\} and \{8, 9, 10\}.

In partitioning, 7 is compared to all numbers. No number from the first set will ever be compared to a number from the second set.

In general, once a pivot \( x, z_i < x < z_j \), is chosen, we know that \( z_i \) and \( z_j \) cannot later be compared.

On the other hand, if \( z_j \) is chosen as pivot before any other element in \( Z_{ij} \) then it is compared to each element in \( Z_{ij} \). Similar for \( z_j \).

In example: 7 and 9 are compared because 7 is first item from \( Z_{7,9} \) to be chosen as pivot, and 2 and 9 are not compared because the first pivot in \( Z_{2,9} \) is 7.

Prior to an element in \( Z_{ij} \) being chosen as pivot, the set \( Z_{ij} \) is together in the same partition ⇒ any element in \( Z_{ij} \) is equally likely to be first element chosen as pivot ⇒ the probability that \( z_i \) or \( z_j \) is chosen first in \( Z_{ij} \) is \( \frac{1}{j-i+1} \)

\[ P_r[z_i \text{ compared to } z_j] = \frac{2}{j-i+1} \]

- We now have:

\[ E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} P_r[z_i \text{ compared to } z_j] \]

\[ = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \]

\[ = \sum_{i=1}^{n} \sum_{k=1}^{n-i} \frac{2}{k} \]

\[ < \sum_{i=1}^{n} O(\log n) \]

\[ = O(n \log n) \]

- Since best case is \( \theta(n \log n) \) ⇒ \( E[X] = \Theta(n \log n) \) and therefore \( E[T(n)] = \Theta(n \log n) \).

Next time we will see how to make quicksort run in worst-case \( O(n \log n) \) time.