Recurrences
(CLRS 4.1-4.2)

- Last time we discussed divide-and-conquer algorithms

<table>
<thead>
<tr>
<th>Divide and Conquer</th>
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</thead>
<tbody>
<tr>
<td>To Solve P:</td>
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<tr>
<td>1. Divide P into smaller problems $P_1, P_2, P_3, ..., P_k$.</td>
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<tr>
<td>2. Conquer by solving the (smaller) subproblems recursively.</td>
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<tr>
<td>3. Combine solutions to $P_1, P_2, ..., P_k$ into solution for P.</td>
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</tbody>
</table>

- Analysis of divide-and-conquer algorithms and in general of recursive algorithms leads to recurrences.

- Merge-sort leads to the recurrence $T(n) = 2T(n/2) + n$
  - or rather, $T(n) = \begin{cases} \Theta(1) & \text{If } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n) & \text{If } n > 1 \end{cases}$
  - but we will often cheat and just solve the simple formula (equivalent to assuming that $n = 2^k$ for some constant $k$, and leaving out base case and constant in $\Theta$).

Methods for solving recurrences

1. Substitution method
2. Iteration method
   - Recursion-tree method
   - (Master method)
1 Solving Recurrences with the Substitution Method

- Idea: Make a guess for the form of the solution and prove by induction.
- Can be used to prove both upper bounds $O()$ and lower bounds $\Omega()$.
- Let’s solve $T(n) = 2T(n/2) + n$ using substitution
  - Guess $T(n) \leq cn \log n$ for some constant $c$ (that is, $T(n) = O(n \log n)$)
  - Proof:
    * Base case: we need to show that our guess holds for some base case (not necessarily $n = 1$, some small $n$ is ok). Ok, since function constant for small constant $n$.
    * Assume holds for $n/2$: $T(n/2) \leq c\frac{n}{2} \log \frac{n}{2}$ (Question: Why not $n - 1$?)
    Prove that holds for $n$: $T(n) \leq cn \log n$

\[
T(n) = 2T(n/2) + n \\
\leq 2(c\frac{n}{2} \log \frac{n}{2}) + n \\
= cn \log \frac{n}{2} + n \\
= cn \log n - cn \log 2 + n \\
= cn \log n - cn + n
\]

So ok if $c \geq 1$
- Similarly it can be shown that $T(n) = \Omega(n \log n)$
  Exercise!
- Similarly it can be shown that $T(n) = T(\lfloor \frac{n}{2} \rfloor) + T(\lceil \frac{n}{2} \rceil) + n$ is $\Theta(n \lg n)$.
  Exercise!
- The hard part of the substitution method is often to make a good guess. How do we make a good (i.e. tight) guess??? Unfortunately, there’s no “recipe” for this one. Try iteratively $O(n^3), \Omega(n^3), O(n^2), \Omega(n^2)$ and so on. Try solving by iteration to get a feeling of the growth.
2 Solving Recurrences with the Iteration/Recursion-tree Method

- In the iteration method we iteratively “unfold” the recurrence until we “see the pattern”.
- The iteration method does not require making a good guess like the substitution method (but it is often more involved than using induction).
- Example: Solve \( T(n) = 8T(n/2) + n^2 \) (\( T(1) = 1 \))

\[
T(n) = n^2 + 8T(n/2) \\
= n^2 + 8(8T(n/4) + (n/2)^2) \\
= n^2 + 8^2T(n/4) + 8(n^2/4) \\
= n^2 + 2n^2 + 8^2T(n/8) \\
= n^2 + 2n^2 + 8^2(8T(n/16) + (n/2)^2) \\
= n^2 + 2n^2 + 8^3T(n/64) + 8^2(n^2/4^2) \\
= n^2 + 2n^2 + 2^2n^2 + 8^3T(n/64) \\
= \ldots \\
= n^2 + 2n^2 + 2^2n^2 + 2^3n^2 + 2^4n^2 + \ldots
\]

- Recursion depth: How long (how many iterations) it takes until the subproblem has constant size? \( i \) times where \( \frac{n}{2^i} = 1 \Rightarrow i = \log n \)
- What is the last term? \( 8^iT(1) = 8^{\log n} \)

\[
T(n) = n^2 + 2n^2 + 2^2n^2 + 2^3n^2 + 2^4n^2 + \ldots + 2^{\log n-1}n^2 + 8^{\log n} \\
= \sum_{k=0}^{\log n-1} 2^k n^2 + 8^{\log n} \\
= n^2 \sum_{k=0}^{\log n-1} 2^k + (2^3)^{\log n}
\]

- Now \( \sum_{k=0}^{\log n-1} 2^k \) is a geometric sum so we have \( \sum_{k=0}^{\log n-1} 2^k = \Theta(2^{\log n-1}) = \Theta(n) \)
- \( (2^3)^{\log n} = (2^{\log n})^3 = n^3 \)

\[
T(n) = n^2 \cdot \Theta(n) + n^3 \\
= \Theta(n^3)
\]
2.1 Recursion tree

A different way to look at the iteration method: is the recursion-tree, discussed in the book (4.2).

- we draw out the recursion tree with cost of single call in each node—running time is sum of costs in all nodes
- if you are careful drawing the recursion tree and summing up the costs, the recursion tree is a direct proof for the solution of the recurrence, just like iteration and substitution
- Example: \( T(n) = 8T(n/2) + n^2 \) \( (T(1) = 1) \)

\[
T(n) = n^2 + 2n^2 + 2^2n^2 + 2^3n^2 + 2^4n^2 + \ldots + 2^\log n - 1 n^2 + 8^\log n
\]
3 Matrix Multiplication

• Let \( X \) and \( Y \) be \( n \times n \) matrices

\[
X = \begin{pmatrix}
  x_{11} & x_{12} & \cdots & x_{1n} \\
  x_{21} & x_{22} & \cdots & x_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{n1} & x_{n2} & \cdots & x_{nn}
\end{pmatrix}
\]

• We want to compute \( Z = X \cdot Y \)

\[ z_{ij} = \sum_{k=1}^{n} X_{ik} \cdot Y_{kj} \]

• Naive method uses \( n^2 \cdot n = \Theta(n^3) \) operations

• Divide-and-conquer solution:

\[
Z = \begin{pmatrix}
  A & B \\
  C & D
\end{pmatrix} \cdot \begin{pmatrix}
  E & F \\
  G & H
\end{pmatrix} = \begin{pmatrix}
  (A \cdot E + B \cdot G) & (A \cdot F + B \cdot H) \\
  (C \cdot E + D \cdot G) & (C \cdot F + D \cdot H)
\end{pmatrix}
\]

– The above naturally leads to divide-and-conquer solution:

  * Divide \( X \) and \( Y \) into 8 sub-matrices \( A, B, C, \) and \( D. \)
  * Do 8 matrix multiplications recursively.
  * Compute \( Z \) by combining results (doing 4 matrix additions).

– Lets assume \( n = 2^c \) for some constant \( c \) and let \( A, B, C \) and \( D \) be \( n/2 \times n/2 \) matrices

  * Running time of algorithm is \( T(n) = 8T(n/2) + \Theta(n^2) \Rightarrow T(n) = \Theta(n^3) \)

– But we already discussed a (simpler/naive) \( O(n^3) \) algorithm! Can we do better?

3.1 Strassen’s Algorithm

• Strassen observed the following:

\[
Z = \begin{pmatrix}
  A & B \\
  C & D
\end{pmatrix} \cdot \begin{pmatrix}
  E & F \\
  G & H
\end{pmatrix} = \begin{pmatrix}
  (S_1 + S_2 - S_4 + S_6) & (S_4 + S_5) \\
  (S_6 + S_7) & (S_2 + S_3 + S_5 - S_7)
\end{pmatrix}
\]

where

\[
\begin{align*}
S_1 &= (B - D) \cdot (G + H) \\
S_2 &= (A + D) \cdot (E + H) \\
S_3 &= (A - C) \cdot (E + F) \\
S_4 &= (A + B) \cdot H \\
S_5 &= A \cdot (F - H) \\
S_6 &= D \cdot (G - E) \\
S_7 &= (C + D) \cdot E
\end{align*}
\]
– Let’s test that $S_6 + S_7$ is really $C \cdot E + D \cdot G$

\[
S_6 + S_7 = D \cdot (G - E) + (C + D) \cdot E
= DG - DE + CE + DE
= DG + CE
\]

– This leads to a divide-and-conquer algorithm with running time $T(n) = 7T(n/2) + \Theta(n^2)$

– We only need to perform 7 multiplications recursively.

– Division/Combination can still be performed in $\Theta(n^2)$ time.

– Let’s solve the recurrence using the iteration method

\[
T(n) = 7T(n/2) + n^2
= n^2 + 7(7T(n/2^2) + (n/2)^2)
= n^2 + (7/2^2)n^2 + 7^2T(n/2^3)
= n^2 + (7/2^2)n^2 + 7^2(7T(n/2^3) + (n/2^2)^2)
= n^2 + (7/2^2)n^2 + (7/2^2)^2n^2 + 7^3T(n/2^3)
= n^2 + (7/2^2)n^2 + (7/2^2)^2n^2 + (7/2^3)^3n^2 .... + (7/2^2)^{\log n - 1}n^2 + 7^{\log n}
= \sum_{i=0}^{\log n - 1} (7/2^2)^i n^2 + 7^{\log n}
= n^2 \cdot \Theta((7/2^2)^{\log n - 1}) + 7^{\log n}
= n^2 \cdot \Theta\left(\frac{7^{\log n}}{n^2}\right) + 7^{\log n}
= n^2 \cdot \Theta\left(\frac{7^{\log n}}{n^2}\right) + 7^{\log n}
= \Theta(7^{\log n})
\]

– Now we have the following:

\[
7^{\log n} = \frac{\log n}{\log 7^2}
= (7^{\log n})^{1 / \log 7^2}
= n^{1 / \log 7^2}
= \frac{\log n}{\log 7^2}
= n^{\log 7}
\]

– Or in general: $a^{\log_k n} = n^{\log_k a}$
So the solution is \( T(n) = \Theta(n^{\log 7}) = \Theta(n^{2.81...}) \)

- Note:
  - We are 'hiding' a much bigger constant in \( \Theta() \) than before.
  - Currently best known bound is \( O(n^{2.376...}) \) (another method).
  - Lower bound is (trivially) \( \Omega(n^2) \).

4 Master Method

- We have solved several recurrences using substitution and iteration.

- We solved several recurrences of the form \( T(n) = aT(n/b) + n^c \) \( (T(1) = 1) \).
  - Strassen's algorithm \( \Rightarrow T(n) = 7T(n/2) + n^2 \) \( (a = 7, b = 2, \text{ and } c = 2) \)
  - Merge-sort \( \Rightarrow T(n) = 2T(n/2) + n \) \( (a = 2, b = 2, \text{ and } c = 1) \).

- It would be nice to have a general solution to the recurrence \( T(n) = aT(n/b) + n^c \).

- We do!

\[
T(n) = aT\left(\frac{n}{b}\right) + n^c \quad a \geq 1, b \geq 1, c > 0
\]

\[
\downarrow
\]

\[
T(n) = \begin{cases} 
\Theta(n^{\log_b a}) & a > b^c \\
\Theta(n^c \log_b n) & a = b^c \\
\Theta(n^c) & a < b^c 
\end{cases}
\]

Proof (Iteration method)

\[
T(n) = aT\left(\frac{n}{b}\right) + n^c
\]

\[
= n^c + a\left(\left(\frac{n}{b}\right)^c + aT\left(\frac{n}{b^2}\right)\right)
\]

\[
= n^c + \left(\frac{a}{b^c}\right) n^c + a^2T\left(\frac{n}{b^2}\right)
\]

\[
= n^c + \left(\frac{a}{b^c}\right) n^c + a^2 \left(\left(\frac{n}{b^2}\right)^c + aT\left(\frac{n}{b^4}\right)\right)
\]

\[
= n^c + \left(\frac{a}{b^c}\right) n^c + \left(\frac{a}{b^c}\right)^2 n^c + a^3T\left(\frac{n}{b^4}\right)
\]

\[
= \ldots
\]

\[
= n^c + \left(\frac{a}{b^c}\right) n^c + \left(\frac{a}{b^c}\right)^2 n^c + \left(\frac{a}{b^c}\right)^3 n^c + \ldots + \left(\frac{a}{b^c}\right)^{\log_b n-1} n^c + a^{\log_b n} T(1)
\]

\[
= n^c \sum_{k=0}^{\log_b n-1} \left(\frac{a}{b^c}\right)^k + a^{\log_b n} n^c
\]

\[
= n^c \sum_{k=0}^{\log_b n-1} \left(\frac{a}{b^c}\right)^k + n^{\log_b a}
\]

Recall geometric sum \( \sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1} = \Theta(x^n) \)
• \( a < b^c \)

\[
a < b^c \iff \frac{a}{b^c} < 1 \Rightarrow \sum_{k=0}^{\log_b n - 1} \left( \frac{a}{b^c} \right)^k \leq \sum_{k=0}^{\infty} \left( \frac{a}{b^c} \right)^k = \frac{1}{1 - \left( \frac{a}{b^c} \right)} = \Theta(1)
\]

\[
a < b^c \iff \log_b a < \log_b b^c = c
\]

\[
T(n) = n^c \sum_{k=0}^{\log_b n - 1} \left( \frac{a}{b^c} \right)^k + n^c \log_b a
\]

\[
= n^c \cdot \Theta(1) + n^c \log_b a = \Theta(n^c)
\]

• \( a = b^c \)

\[
a = b^c \iff \frac{a}{b^c} = 1 \Rightarrow \sum_{k=0}^{\log_b n - 1} \left( \frac{a}{b^c} \right)^k = \sum_{k=0}^{\log_b n - 1} 1 = \Theta(\log_b n)
\]

\[
a = b^c \iff \log_b a = \log_b b^c = c
\]

\[
T(n) = \sum_{k=0}^{\log_b n - 1} \left( \frac{a}{b^c} \right)^k + n^c \log_b a
\]

\[
= n^c \Theta(\log_b n) + n^c \log_b a = \Theta(n^c \log_b n)
\]

• \( a > b^c \)

\[
a > b^c \iff \frac{a}{b^c} > 1 \Rightarrow \sum_{k=0}^{\log_b n - 1} \left( \frac{a}{b^c} \right)^k = \Theta \left( \left( \frac{a}{b^c} \right)^{\log_b n} \right) = \Theta \left( \frac{a^{\log_b n}}{b^c^{\log_b n}} \right) = \Theta \left( \frac{a^{\log_b n}}{n^c} \right)
\]

\[
T(n) = n^c \cdot \Theta \left( \frac{a^{\log_b n}}{n^c} \right) + n^c \log_b a
\]

\[
= \Theta(n^{\log_b a}) + n^c \log_b a = \Theta(n^{\log_b a})
\]

Note: Book states and proves the result slightly differently (don’t read it).

5 Changing variables

Sometimes recurrences can be reduced to simpler ones by changing variables

• Example: Solve \( T(n) = 2T(\sqrt{n}) + \log n \)

Let \( m = \log n \Rightarrow 2^m = n \Rightarrow \sqrt{n} = 2^{m/2} \)

\( T(n) = 2T(\sqrt{n}) + \log n \Rightarrow T(2^m) = 2T(2^{m/2}) + m \)

Let \( S(m) = T(2^m) \)

\( T(2^m) = 2T(2^{m/2}) + m \Rightarrow S(m) = 2S(m/2) + m \)

\( \Rightarrow S(m) = O(m \log m) \)

\( \Rightarrow T(n) = T(2^m) = S(m) = O(m \log m) = O(\log n \log \log n) \)
6 Other recurrences

Some important/typical bounds on recurrences not covered by master method:

- **Logarithmic**: $\Theta(\log n)$
  - Recurrence: $T(n) = 1 + T(n/2)$
  - Typical example: Recurse on half the input (and throw half away)
  - Variations: $T(n) = 1 + T(99n/100)$

- **Linear**: $\Theta(n)$
  - Recurrence: $T(n) = 1 + T(n - 1)$
  - Typical example: Single loop
  - Variations: $T(n) = 1 + 2T(n/2), T(n) = n + T(n/2), T(n) = T(n/5) + T(7n/10 + 6) + n$

- **Quadratic**: $\Theta(n^2)$
  - Recurrence: $T(n) = n + T(n - 1)$
  - Typical example: Nested loops

- **Exponential**: $\Theta(2^n)$
  - Recurrence: $T(n) = 2T(n - 1)$