Lecture 7: Sorting Lower Bound and Radix-Sort

(CLRS 8.1-8.3)

May 24th, 2002

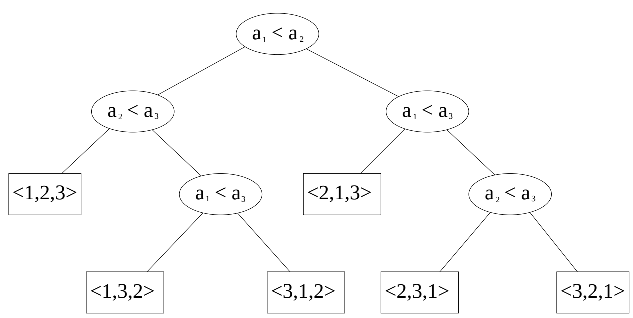
1 Comparison model sorting lower bound

- We have seen two $\Theta(n \log n)$ sorting algorithms: Merge-sort and quick-sort (using median selection)
- These algorithms only use comparisons to gain information about the input.
- We will prove that such algorithms have to do $\Omega(n \log n)$ comparisons
- To prove bound, we need *formal model*

Decision tree

- Binary tree where each internal node is labeled $a_i \leq a_j$ (a_i is the i'th input element)
- Execution corresponds to root-leaf path
  - at each internal node comparisons $a_i \leq a_j$ is performed and branching made
- Leaf contains result of computation

- Example: Decision tree for sorting 3 elements.



  - a leaf contains permutation giving sorted order.

- Note: Decision tree model corresponds to algorithms where
  - Only comparisons can be used to gain knowledge about input
  - Data movement, control, etc, are ignored
- Worst case number of comparisons performed corresponds to maximal height of tree $\Rightarrow$ lower bound on height $\Rightarrow$ lower bound on sorting

Theorem: Any decision tree sorting n elements has height $\Omega(n \log n)$

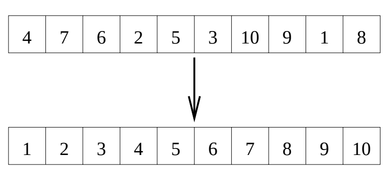
Proof:

- Assume elements are the (distinct) numbers 1 through n
- There must be $n!$ leaves (one for each of the $n!$ permutations of n elements)
- Tree of height h has at most 2^h leaves

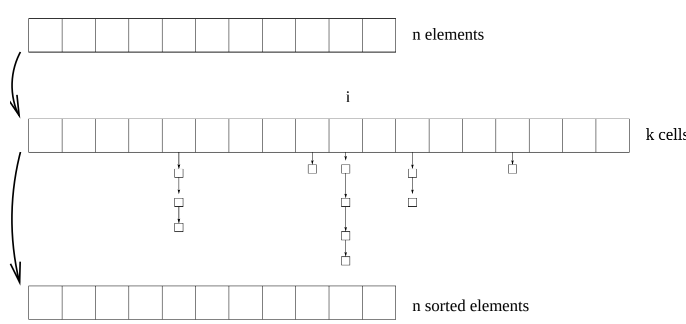
$$
\begin{aligned}
2^h \geq n! \Rightarrow h &\geq \log(n!) \\
&= \log(n(n-1)(n-2)\cdots(2)) \\
&= \log n + \log(n-1) + \log(n-2) + \cdots + \log 2 \\
&= \sum_{i=2}^{n} \log i \\
&= \sum_{i=2}^{n/2-1} \log i + \sum_{i=n/2}^{n} \log i \\
&\geq 0 + \sum_{i=n/2}^{n} \log \frac{n}{2} \\
&= \frac{n}{2} \cdot \log \frac{n}{2} \\
&= \Omega(n \log n)
\end{aligned}
$$

2 Beating sorting lower bound (bucket sort)

- While proving the $\Omega(n \log n)$ comparison lower bound we assumed that the input were integers 1 through n
- We can easily sort integers 1 through n in $O(n)$ time.
  - just move element i to position i in output array



- What about the more general problem of sorting n elements in range $1....k$?
  - Move element i to linked list of element i
  - Produce sorted output



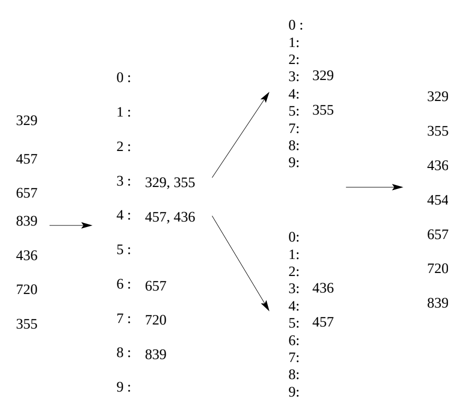
- Algorithm uses $O(n+k)$ time and space
- Note:
  - We did not use comparison at all!
  - We beat the $\Omega(n \log n)$ bound by using values of elements to index into array—*Indirect addressing*
- Note:
  - Algorithm is *stable* (Order of equal elements maintained)
  - Algorithm is not *in-place* (more than $O(n)$ space use)—All other sorting algorithms we have seen have been in-place
- Note:
  - Book calls the algorithm (or simplified version of it) *counting sort* and use *bucket sort* for something else
  - I call it *bucket sort* (we put elements in buckets)

3 Radix Sort

- Problem with bucket sort is that k can be very large
  - Example: 32 bit integers $\Rightarrow k = 2^{32} \approx 10^9 \Rightarrow$ space used is $10^9 \cdot 4$ bytes $\approx$ 4Gbytes!
- Large k result in running time not proportional to n (and other problems like disk swapping)

3.1 MSD Radix-sort

- MSD Radix-sort regards numbers as being made up of digits
  - Bucket sort by most significant digit (MSD)
  - Recursively sort buckets with more than one element (according to next digit)
- Correctness is straightforward (Induction)
- Example: Sorting numbers < 1000 ($k = 1000$) using 10 buckets



- Problem with MSD radix sort
  - We need to keep track of a lot of recursion (buckets)
  - Many buckets $\Rightarrow$ space use
- Advantages of MSD radix sort
  - We only need to look at *distinguishing prefix* (what we need to look at)

3.2 LSD Radix-sort

- LSD Radix-sort:
  - Sort by least significant digit (LSD)
  - Sort by second least significant digit (using a stable sorting algorithm)

    $\vdots$
  - Sort by most significant digit (using a stable sorting algorithm)
- Correctness again by induction
- Example:

329		0 : 720		720		0 :		720	0 :	329
457		1 :		355		1 :		329	1 :	355
657		2 :		436		2 : 720,329		436	2 :	436
839	$\longrightarrow$	3 :	$\longrightarrow$	457	$\longrightarrow$	3 : 436,839	$\longrightarrow$	839	3 : 329,355	454
436		4 :		657		4 :		355	$\longrightarrow$ 4 : 436,457 $\longrightarrow$	657
720		5 : 355		329		5 : 355,457,657		457	5 :	720
355		6 : 436		839		6 :		657	6 : 657	839
		7 : 457,657				7 :			7 : 720	
		8 :				8 :			8 : 839	
		9 : 329,839				9 :			9 :	

- Problems with LSD Radix-sort:
  - We look at all the numbers in all phases
  - Not generally in-place ($n < 10$)

3.3 In-place Radix-sort

- To get in-place algorithm we simply choose number of buckets equal n in radix sort
  - In example, we had $n = 7$ and 10 buckets
- When doing so we divide the numbers in ranges of n
  - In example, we divided in ranges of 10
- If numbers are $\leq R$ the number of phases i is $n^i = R \Rightarrow i = \frac{\log R}{\log n}$
  - In example, we had $R = 839$, $10^3 > 839 \Rightarrow 3$ phases

$\Downarrow$

- $O(n)$ space and $O(n \cdot \frac{\log R}{\log n})$ time

- Note: When is in-place Radix-sort better than $1 \cdot n \log n$ sort (for 32 bit integers)?
  - $n \cdot \frac{32}{\log n} < n \log n \Rightarrow \log^2 n > 32 \Rightarrow n > 2^{\sqrt{32}}$
  - $2^{\sqrt{32}} < 2^6 = 64$
- Note: Recent algorithm by Anderson et al. (1997) combines advantages of MSD and LSD radix sort
  - In-place
  - Only look at distinguishing prefix