Algorithms
Computer Science 140 & Mathematics 168
Instructor: B. Thom
Fall 2004
Homework 1a
Due on Thursday, 09/02/04 (beginning of class)

1. **[15 points] Properties of Logs.**
   Let's begin by proving that $\log_b xy = \log_b x + \log_b y$. The proof goes as follows: Let $k = \log_b xy$, $\ell = \log_b x$, and $m = \log_b y$. Then, $b^k = xy$, $b^\ell = x$, and $b^m = y$, by the definition of the logarithm. Thus, $b^k = b^\ell b^m = b^{\ell+m}$ by properties of exponents. Thus, $k = \ell + m$, which is what we had set out to prove.

   Now give proofs for each of the following properties of logarithms. Write your proofs out carefully. You should assume that a, b, c, n are positive *real numbers* (not necessarily integers).

   (a) $\log_b a^n = n \log_b a$.
   (b) $\log_b a = \frac{\log_c a}{\log_c b}$.
   (c) $a^{\log_b n} = n^{\log_b a}$.

2. **[10 points] Growth of Functions.** In this problem you will be doing two proofs by induction. Please write your proofs clearly, with a basis, induction hypothesis, and induction step.

   (a) Use induction to show that $n^2 \geq 2n + 1$ for $n \geq 3$.
   (b) Now use induction to show that $2^n \geq n^2$ for $n \geq 5$. (Using the result from part (a) of this problem will be most useful in doing this proof correctly!)

3. **[15 points] Extra Credit (due Friday at 6pm): Arithmetic and Geometric Meanings.** Let $x_1, \ldots, x_n$ be positive real numbers. The *arithmetic mean* of these numbers is defined to be $\frac{x_1+x_2+\ldots+x_n}{n}$ and the *geometric mean* is defined to be $(x_1 x_2 \cdots x_n)^{1/n}$. In this problem we show that the arithmetic mean of n numbers is at least as large as the geometric mean of those numbers.

   (a) Use induction to show that if $x_1 x_2 \cdots x_n = 1$ then $x_1 + x_2 + \ldots + x_n \geq n$. Observe that this is a special case of the statement we are trying to prove.
   (b) Use this fact to show that the arithmetic mean is at least as large as the geometric mean. (No induction required here; just a little algebra.)