Simplified External Memory Algorithms for Planar DAGs

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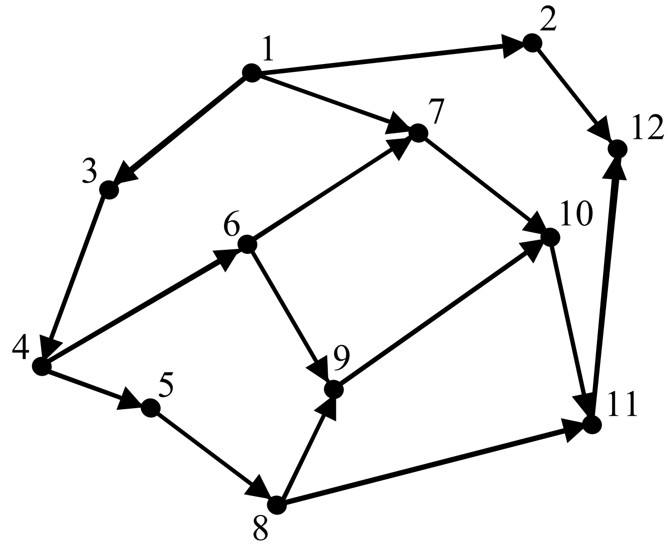
Graph Problems

- Graph G = (V, E) with V vertices and E edges
  - DAG: directed acyclic graph
  - G is planar if it can be drawn in the plane so no edges cross

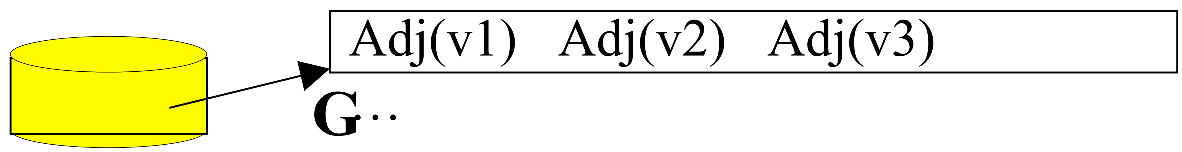
- Some fundamental problems:
  - BFS, DFS
  - Single-source shortest path (SSSP)
  - Topological order of a DAG
    - A labeling of vertices such that if (v,u) in E then $\mu(v) < \mu(u)$



Massive graphs

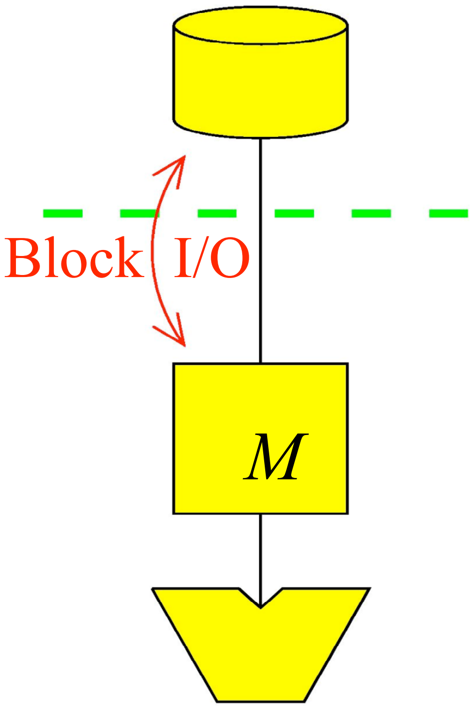
- Massive planar graphs appear frequently in GIS
  - Terrains are stored as grids or triangulations
  - Example: modeling flow on terrain
    - Each point is assigned a flow direction such that the resulting graph is directed and acyclic
    - To trace the amount of flow must topologically sort this graph

- Massive graphs stored on disk
  - Assume edge-list representation stored on disk



- I/O can be severe bottleneck

I/O Model [AV'88]



- Parameters:

  N = V+E

  B = disk block size

  M = memory size

- I/O-operation:
  - movement of one block of data from/to disk

- Fundamental bounds:

  Scanning: $\text{scan}(N) = O(\frac{N}{B})$ I/Os

  Sorting: $\text{sort}(N) = O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ I/Os

- In practice B and M are big

$$\frac{N}{B} < \frac{N}{B}\log_{M/B}\frac{N}{B} << N$$

Previous Results

- **Lower bound:** $\Omega(\min\{V, \text{sort}(V)\})$ (practically sort(V))
- Not matched for most general graph problems, e.g.
  - **General undirected graphs**
    - BFS: $O(\sqrt{\frac{VE}{B}} + \text{sort}(E))$ [MM'02]
    - SSSP: $O(\sqrt{\frac{VE}{B}\log\frac{W}{w}} + \text{sort}(E))$ [MZ'03]
    - DFS: $O((V+\frac{E}{B})\log V + \text{sort}(E))$ [KS'96]
  - **General directed graphs**
    - BFS, DFS, SSSP, topological order (DAG)

      $O((V+\frac{E}{B})\log V + \text{sort}(E))$ [BVWB'00]
- Sparse graphs *E=O(V)*
  - Directed BFS, DFS, SSSP: O(V) I/Os

Previous Results

- Improved algorithms for special classes of (sparse) graphs
  - Planar undirected graphs solved using
    - $O(\text{sort}(N))$ reductions

    - $O(\text{sort}(N))$ multi-way separation algorithm [MZ'02]

  - Generalized to planar directed graphs
    - BFS, SSSP: $O(\text{sort}(N))$ [ATZ'03]
    - DFS: $O(\text{sort}(N) \log N)$ [AZ'03]
    - Planar DAG topological sort : $O(\text{sort}(N))$ [ATZ'03]
      - Computed using a DED of the dual graph

Our Results

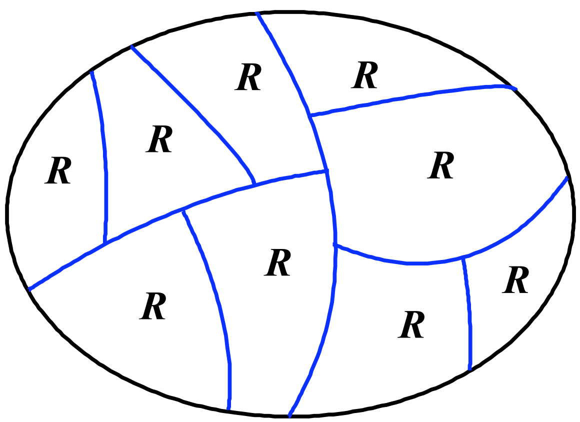
- Simplified algorithms for planar DAGs

O(scan(N)) I/Os
separation =============> top order, BFS, SSSP

- This does not improve the O(sort(N)) known upper bound since computing the separation takes O(sort(N)) I//Os
- Previous algorithms take O(sort(N)) even if separation is given

Planar graph separation: R-partition

- A partition of a planar graph using a set ***S*** of separator vertices into $O(\frac{N}{R})$ subgraphs (clusters) of at most R vertices each such that:

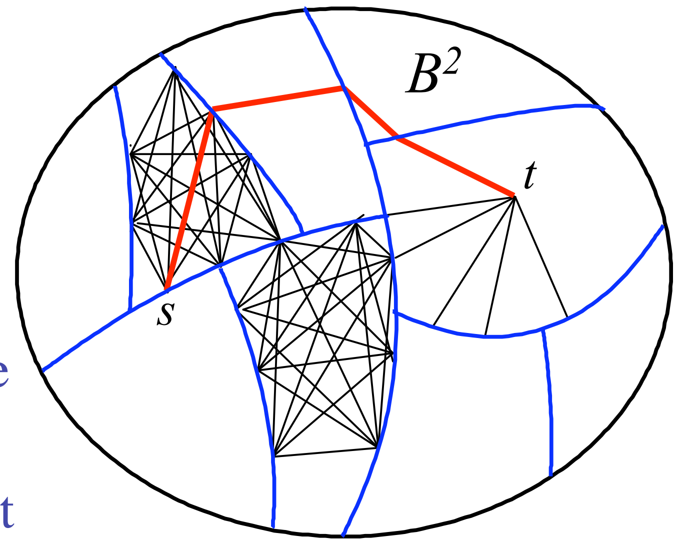


$O(\frac{N}{\sqrt{R}})$ separators vertices in total

Each cluster is adjacent to $O(\sqrt{R})$ separator vertices

- In external memory choose $R = B^2$
  - $O(N/B)$ separator vertices
  - $O(N/B^2)$ clusters, $O(B^2)$ vertices each and $O(B)$ boundary vertices
  - Can be computed in $O(\text{sor}(N))$ I/Os [MZ'02]

Planar SSSP



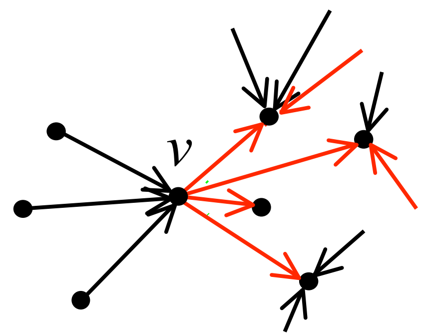
1. Compute a B^2-partition of G
2. Construct a substitute graph G^R on the separator vertices such that it preserves SP in G between any u,v in S
   – replace each subgraph G_i with a complete graph on boundary of $\boldsymbol{G_i}$
   – for any u, v on boundary of G_i, the weight of edge (u,v) is $\delta_{G_i}(\boldsymbol{u,v})$
3. Solve SSSP on $\boldsymbol{G^R}$
4. Find SSSP to vertices inside clusters

Computed efficiently using

- $\boldsymbol{G^R}$ has $O(N/B)$ vertices and $O(N)$ edges
- Properties of the B^2-partition

A Topological Sort Algorithm

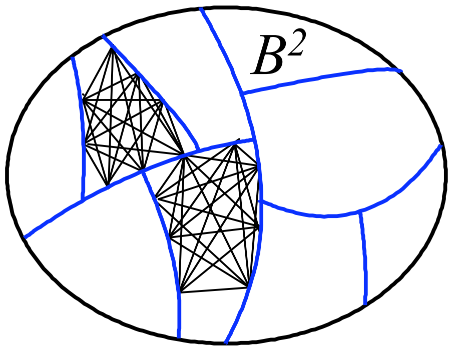
- Compute indegree of each vertex
- Maintain list Z of indegree-zero vertices
- Repeatedly
  - Number an indegree-zero vertex v
  - Consider all edges (v,u) and decrement indegree of u
  - If indegree(u) becomes 0 insert u in list Z



- O(1) I/O per edge ==> O(N) I/Os

Topological Sort using B^2-partition

1. **Construct a substitute graph G^R using B^2-partition**
   - edge from v to u on boundary of G_i iff exists path from v to u in G_i



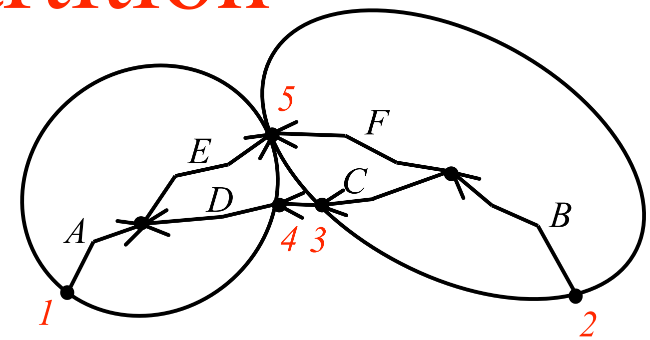
- *Lemma: for any separator vertices u,v if u is reachable from v in G, then u is reachable from v in G^R*

2. Topologically sort G^R (separator vertices in G)

- *Lemma: A topological order on G^R is a topological order on G.*

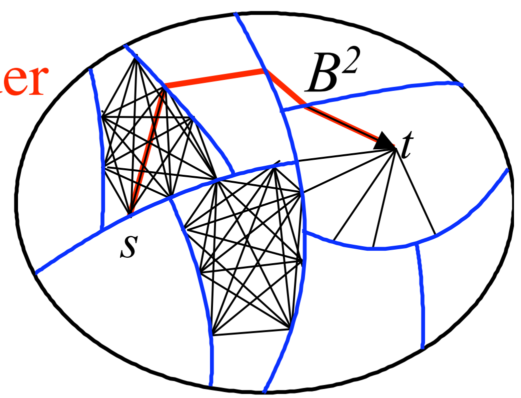
3. Compute topological order inside clusters

Topological Sort using B^2-partition



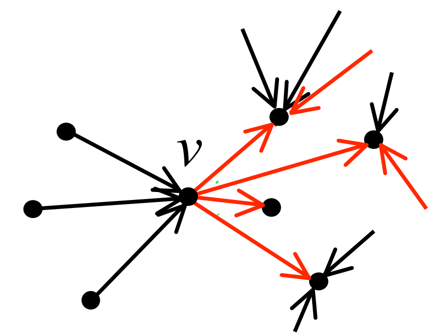
- Problem:
  - Not clear how to incorporate removed vertices from G in topological order of separator vertices (G^R)

- Solution (assuming only one in-degree zero vertex *s* for simplicity):
  - Longest-path-from-*s* order is a topological order
  - Longest paths to removed vertices locally computable from longest-paths to boundary vertices



Topological Sort using B^2-partition

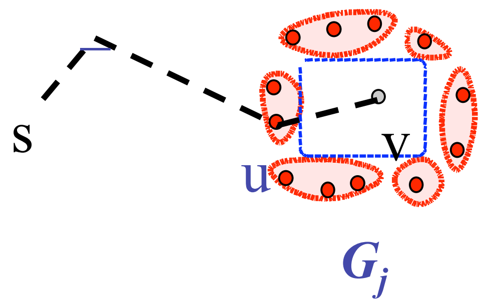
1. Construct substitute graph G^R
   - Weight of edge between v and u on boundary of G_i equal to length of longest path from v to u in G_i
2. Topologically sort G^R
3. Compute longest path to each vertex in G^R (same as in G):
   - Maintain list L of longest paths seen to each vertex
   - Repeatedly:
     - Obtain longest path for the next vertex v in topological order
     - Consider all edges (v,u) and update longest path to u in L



4. Find longest path to vertices inside clusters

Longest path to vertices inside a cluster

- must cross the boundary of the cluster

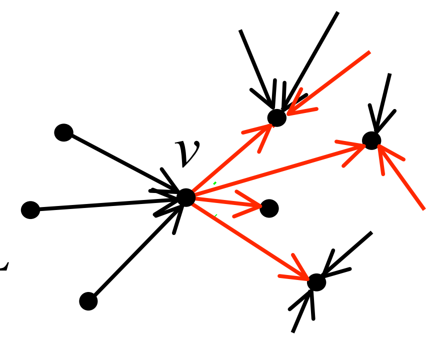


$$LP(v) = \max_{u \in Bnd(Gi)} \{LP(u) + LP_{Gi}(u,v)\}$$

Analysis

Topologically sort G^R

- Maintain a list L with the indegree of each vertex
- Maintain list Z of indegree-zero vertices
- Repeatedly
  - Number an indegree-zero vertex v from Z
  - Consider all edges (v,u) and decrement indegree of u in L
  - If indegree(u) becomes 0 insert u in list Z



- G^R has $O(N/B)$ vertices and $O(N/B^2 \times B^2) = O(N)$ edges
- Each vertex: access its adjacency list ==> O(N/B) I/Os
- Each edge (v,u): update L ==> O(N) I/Os
  - Can be reduced to O(N/B) using boundary set property

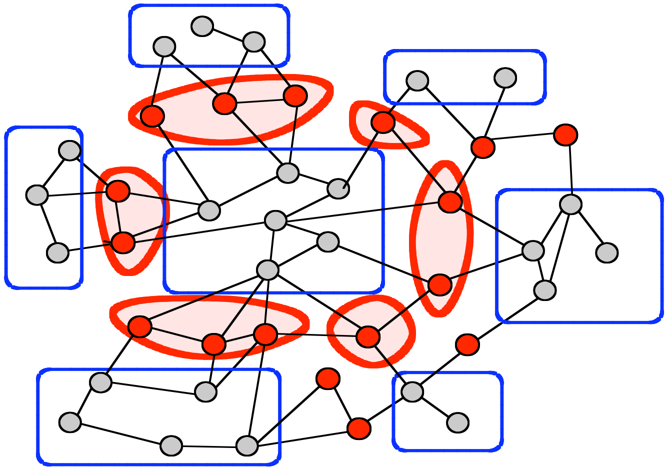
Analysis

- Boundary set in the B^2-partition

  (Maximal) set of separator vertices adjacent to the same clusters

  – A boundary set is of size O(B)

  – There are $O(N/B^2)$ boundary sets [F'87] (ass. bounded degree)



- We store L so that vertices in the same boundary set are consecutive

  – Vertices in same boundary set have same $O(B)$ neighbors in G^R

  – Each boundary set is accessed once by each neighbor in G^R

  – Each boundary set has size $O(B)$

➔ $O(N/B^2) \times O(B) = O(N/B)$ I/Os

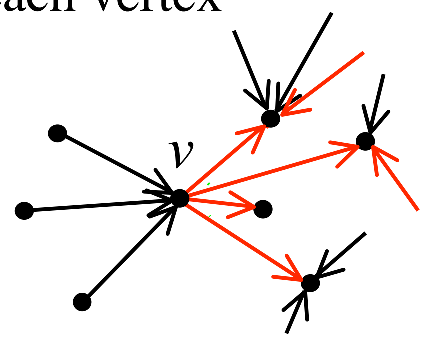
Conclusion and Open Problems

- Given a B^2-partition topological order can be computed in O(scan)N)) I/Os
- Longest path idea can also be used to compute SSSP and BFS in the same bound
- Open problems:
  - Improved DFS on DAGs? (exploiting acyclicity)
    - Planar directed DFS O(sort(N) log N)

Analysis

Compute longest path to each vertex in G^R (same as in G):

- Maintain list L' of longest paths seen to each vertex
- Repeatedly:
  - Obtain longest path for next vertex v in topological order
  - Consider all edges (v,u) and update longest path to u



- We store L' so that vertices in the same boundary set are consecutive
  - Vertices in same boundary set have same $O(B)$ neighbors in G^R
  - Each boundary set is accessed once by each neighbor in G^R
  - Each boundary set has size $O(B)$

➔ $O(N/B^2) \times O(B) = O(N/B)$ I/Os