

The Mathematics
of
Climate Change

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- Climate Change is a New Phenomenon
- Notoriously difficult to model mathematically
- Science is New and Uncertain

For the first time in recorded history
humans can change the atmosphere of the planet,
its water bodies,
and the complex web of species that makes life on Earth.

- While we create new mathematics
- And figure out how to act
- Time is running out

What actions to take?

- The situation is rife with Uncertainty
- Unknown Probabilities, Parameters & Equations
- Even the scope of the Risks is unknown
- Science is Uncertain

We know

- Knowledge is Imperfect & Limited
- Time is short
- Outcome could be Catastrophic
- Risks are real

- Can Mathematics help under these conditions?

Yes

Decisions under uncertainty

with catastrophic risks

- Requires new mathematical developments
- New axiomatic approach and new mathematical techniques
- To make decisions and take action under extreme uncertainty, facing rare long run and catastrophic events

Without new tools

- We are misled & paralyzed into inaction

Scientific Background

How do we currently evaluate actions involving projects with uncertain costs and benefits, over time?

‘Expected utility analysis’

- Emerged from Von Neumann Morgenstern Axioms, studied by Milnor and Arrow.
- By law, the procedure must be used by US Congress before funding large projects.

Examples: Asteroids and Global Warming

September 16 2000: the British Task Force on Potentially Hazardous Near Earth Objects (NEO) Report classified asteroids risks by their size, energy yield, and average interval between impacts.

Large mass extinctions - e.g. the Cretaceous Tertiary Geological boundary - follow from the impact of asteroids of 16 km in diameter.

They occur on average once every 100 million years, and threaten the survival of advanced life forms on the planet.

Asteroids are a well known threat to the survival of the human species.

Risk profile:

- An asteroid impact of this magnitude occurs on average once every 100 million years
- Damages are \$120 trillion, obliterating all human produced value in the planet
- Damage is permanent - it continues for about 1 billion years, the expected lifetime of our planet before the sun becomes a red star
- Existing observations: such an event will not happen in the next 30 years

Comparison

Risk of Global Warming:

- Probability of global warming is 1, it is happening
- Will produce a permanent loss of about \$2 trillion a year in the US and globally about \$8 trillion (Posner, 2004).
- No consensus on whether the gradual or the catastrophic case is more likely.

Evaluating Asteroid Impacts

A standard evaluation: present expected value, which brings future uncertain value into the present.

- The expected loss per year is \$120 trillion or $\$120 \times 10^{12}$ - times the probability of the loss, which occurs on average once every 100 million years, or 10^{-8} . The loss is permanent. Therefore in year N the loss is

$$\begin{aligned} & \sum_{t=1}^{\infty} (120 \times 10^{12} \times 10^{-8}) \cdot \delta^{N+t} \\ &= (120 \times 10^{12} \times 10^{-8} \times \delta^N) \sum_{t=1}^{\infty} \delta^t \\ &= (120 \times 10^{12} \times 10^{-8} \times \delta^N) \left(\frac{\delta}{1 - \delta} \right) \end{aligned}$$

δ is the time 'discount factor', $0 < \delta < 1$

$1 - \delta$ the 'discount rate'

If the risk does not occur on year N , it can occur

on year $N + 1$, $N + 2$, $N + 3$, etc. Each time it occurs, it is permanent. Thus the total risk is the sum of the risk on year 30, plus the risk on year 31, plus the risk on year 32:

$$\begin{aligned} & (120 \times 10^{12} \times 10^{-8} \times \delta^N) \left(\frac{\delta}{1 - \delta} \right) \sum_{j=1}^{\infty} \delta^j = \\ & = (120 \times 10^{12} \times 10^{-8} \times \delta^N) \left(\frac{\delta}{1 - \delta} \right)^2 \end{aligned}$$

At a 5% discount rate $\delta = 0.95$, total expected discounted value is

$$\begin{aligned} & 120 \times 10^{12} \times 10^{-8} \times \frac{95}{100}^{30} \times \left(\frac{\frac{95}{100}}{1 - \frac{95}{100}} \right)^2 = \\ & = 9.2982 \times 10^7 \sim \$92 \text{ million.} \end{aligned}$$

At a 10% discount rate

$$120 \times 10^{12} \times 10^{-8} \times \frac{90}{100}^{30} \times 81 = 4.1204 \times 10^6 \sim \$4 \text{ million.}$$

At 3%

$$\begin{aligned} & 120 \times 10^{12} \times 10^{-8} \times \frac{97}{100}^{30} \times 32^2 = \\ & = 4.9276 \times 10^8 \sim \$500 \text{ million} \end{aligned}$$

Evaluating global warming

The Intergovernmental Panel on Climate Change (IPCC) finds that human - induced global warming is already occurring. Two main scenarios: (1) catastrophic global warming, and (2) more gradual build up of damages.

- **Scenario 1: Catastrophic global warming.** A rapid increase in temperature which produces a permanent loss of \$2 trillion a year in the US and globally \$8 trillion. The present discounted value at a 3% discount rate is

$$\begin{aligned} 2 \times 10^{12} \times \frac{\delta}{1 - \delta} &= 2 \times 10^{12} \times \frac{97}{3} = \\ &= 6.4667 \times 10^{13} \sim \$65 \text{ trillion} \end{aligned}$$

At a 5%

$$2 \times 10^{12} \times \frac{95}{5} =: 3.8 \times 10^{13} \sim \$38 \text{ trillion}$$

and at 10%

$$2 \times 10^{12} \times 9 =: 1.8 \times 10^{13} \sim \$18 \text{ trillion}$$

Scenario 2: Gradual build up of damages from global warming.

Temperature increases slowly and damages increase for about 100 years to reach 1% of the planet's GDP, which is \$120 trillion . After we reach maximum, either (1) annual damage remains *a perpetuity*, or (2) damages decrease slowly and disappear in 100 years.

Present discounted value with 3% discount rate is

$$\sum_{i=1}^{\infty} \frac{10^{12}}{100} \times i \times \frac{97}{100}^i = 1.0778 \times 10^{13} \sim \$10 \text{ trillion}$$

with 5%

$$\sum_{i=1}^{\infty} \frac{10^{12}}{100} \times i \times \frac{95}{100}^i = 3.8 \times 10^{12} \sim \$3.8 \text{ trillion}$$

and with 10%

$$\sum_{i=1}^{\infty} \frac{10^{12}}{100} \times i \times \frac{9}{10}^i = 9.0 \times 10^{11} \sim \$900 \text{ billion}$$

In the second case, when the damage gradually decreases with a 3% discount rate

$$\left[\sum_{i=1}^{100} \frac{10^{12}}{100} \times i \times \frac{97}{100}^i \right] + \left[\sum_{i=2}^{100} \frac{10^{12}}{100} \times (100 - (i-1)) \times \frac{97}{100}^{100+i} \right] =$$

$$9.7456 \times 10^{12} \sim \$9.7 \text{ trillion}$$

with a 5% discount rate

$$\left[\sum_{i=1}^{100} \frac{10^{12}}{100} \times i \times \frac{95}{100}^i \right] + \left[\sum_{i=2}^{100} \frac{10^{12}}{100} \times (100 - (i-1)) \times \frac{95}{100}^{100+i} \right] =$$
$$3.7506 \times 10^{12} \sim \$3.7 \text{ trillion}$$

and with a 10% discount

$$\left[\sum_{i=1}^{100} \frac{10^{12}}{100} \times i \times \frac{9}{10}^i \right] + \left[\sum_{i=2}^{100} \frac{10^{12}}{100} \times (100 - (i-1)) \times \frac{9}{10}^{100+i} \right] =$$
$$8.9993 \times 10^{11} \sim \$890 \text{ billion.}$$

In all cases, global warming overwhelms in terms of present discounted value an asteroid impact, even though in the non catastrophic case global warming decreased GDP by only 1% and only after 100 years.

Asteroid Impacts v. Global Warming

Major disturbances caused by global warming - even with conservative losses of less than 1% of GDP building up over 100 years - overwhelm the costs of an asteroid impact which can lead to the extinction of the human species?

How can this be?

The expected loss from an asteroid that leads to extinction is between \$500 million and \$92 billion, smaller than a failure of a single atomic plant - the Russians lost \$140 billion at Chernobyl - or the risks involved in global warming – between \$890 billion and \$9.7 trillion.

Using expected values we are led to believe that preventing asteroid impacts should not rank high in our policy priorities.

Common sense rebels against the computation we just provided. Lets use some examples for comparison.

- In 2004, profits of the 10 biggest oil companies were \$100 billion. It seems unreasonable that losses from asteroid impacts are between \$500 million and \$92 billion.

It seems difficult to believe that we will avert human extinction, since current priorities will always outweigh infrequent events, no matter how important they may be.

- Is there anything wrong with this argument?
- The following provides an alternative.

Catastrophes and the Survival of the Species

Posner (2004) classifies **catastrophes** into various types, all rare events threatening the survival of our species.

He argues that *expected present value* does not capture the true impact of a catastrophe, and that something else is at stake.

Because of his loyalty to the concept of expected present value, he argues that 'rationality' does not work for catastrophes, and we cannot deal rationally with small probabilities events that cause such large and irreversible damage.

A similar issue is raised in Behavioral Finance with respect to the various so called 'paradoxes' including EPP, Allais Paradox, and the Fixed Income Return Paradox.

The problem is NOT rationality.

There may be a *different rational* perspective needed when considering the long range future of the species, or catastrophes.

Expected value is a good measure for evaluating risks that have a good chance to occur in our lifetime, but not for evaluating risks that are important but have essentially a zero chance to occur while we are alive.

For such risks we may need another approach.

In our current state of evolution we may oppose a human tendency to give preference to immediate outcomes as opposed to more distant and dangerous ones.

Chichilnisky (1994, 2000, 2006,7,8) re-considers the foundations of the present discounted value: VNM's axioms for choice under uncertainty.

Plan of Action

Step 1: Show why VNM axioms are 'biased' against the future, Chichilnisky, 1994, 2000.

Discounted utility is biased against the future – favors projects that have high returns today against those who have higher returns in the future, a bias against many important long term environmental issues such as global warming.

Step 2: Define new axioms for time choices: require **symmetry** in the treatment of the present and the future, 1994, 2000

Step 2: Define new axioms for rare events 2006.

Step 3. Show new axioms *coincide* with Von Neumann and Morgenstern's with 'normal' events, e.g. those likely to occur in our lifetime (2006).

Step 4: Representation Theorem identifying all rankings that satisfy the new axioms over time and uncertainty (2000, 2006)

Step 5: Comparison with VNM's axioms: critical role of 'Monotone Continuity Axiom' and choice of 'Topology'. Our Axiom 2 logically negates MCA: The Topology of Fear (2006).

Step 6: Experimental Evidence - Establishes New Axioms when Subjects face Fear (2007, 2008)

- The two sets of axioms are consistent with each other for 'normal' events but are quite different on events involving a long run future or catastrophic events.
- How can this be?

Example

Classical Mechanics and General Relativity

Classic mechanics applies to 'normal scales' closer to our daily reality. Relativity applies to large scale phenomena involving astral bodies.

Both are correct in their respective scales. Neither contradicts the other.

The same is the case with the Von Neumann Morgenstern axioms and the new axioms.

- Rare Events and the Long Run Future - same mathematical objects

How?

- **Rare events** are really small probability events
- **Long Run Future events** become **rare** when discounted

New Axioms:

The Present & the Long Run Future

- (i) No dictatorship of the present
- (ii) No dictatorship of the future
- (iii) Continuity in L_∞ and Linearity, a standard condition.

New Axioms

Frequent & Rare Events

- (i) No dictatorship of frequent events
- (ii) No dictatorship of rare events
- (iii) Continuity in L_∞ and Linearity, a standard condition.

For each $t = 1, 2, \dots, \alpha_t \in R$ represents the utility of a state - or a generation - t .

$\alpha = \alpha_t$ and $\beta = \beta_t$, are paths of utility across states - or time.

Paths are bounded: $\alpha, \beta \in L_\infty$.

The ranking of paths - or lotteries - is represented by a real valued function $W : L_\infty \longrightarrow R$.

A ranking is a **'dictatorship of the present'** when for every α, β

$$W(\alpha) > W(\beta) \Leftrightarrow W(\alpha') > W(\beta')$$

where α' and β' are arbitrary modifications of α and β beyond a period $T = T(\alpha, \beta)$.

Axiom (i) rules out dictatorship of the present.

A ranking is a **'dictatorship of the future'** when

$$W(\alpha) > W(\beta) \Leftrightarrow W(\alpha') \geq W(\beta')$$

where α' and β' are obtained by modifying arbitrarily α and β in any finite number of periods.

Axioms (ii) rules out dictatorships of the future.

Examples

1. Discounted utility functions

$$W(\alpha) = \sum_{t=1}^{\infty} \lambda^{-t} \alpha_t$$

are dictatorships of the present. They are ruled out. Indeed:

Lemma 1 (Chichilnisky, 1992, 1996): Von Neumann-Morgenstern Axioms lead to dictatorships of the present.

Proof:

Let $d\mu(s)$ denote $e^{-\mu t} dt$ as in proper discounted utility – or more generally any other L_1 measure used for discounting utility over time.

Then

$$\int_R u(x(t))d\mu(t) > \int_R u(y(t))d\mu(t) \Leftrightarrow \exists \delta > 0 :$$

$$\int_R u(x(t))d\mu(t) > \int_R u(y(t))d\mu(t) + \delta.$$

Now let $\varepsilon = \delta/6K$ where

$$K = \sup_{x \in L, s \in R} |u(x(t))|$$

and recall that $u \in L_\infty$.

Then if a path $x' > x$ and another $y' = y$ a.e. on any set S^c where $\mu(S) < \varepsilon$,

$$\begin{aligned} & \left| \int_R u(x'(t))d\mu(t) - \int_R u(x(t))d\mu(t) \right| \\ & \leq 2K\mu(t) < 2K.\delta/6K = \delta/3. \end{aligned}$$

Therefore $y \prec x \Rightarrow \int_R u(y'(t))d\mu(t) < \int_R u(x'(t))d\mu(t)$

so that

$$y' \prec x' \text{ i.e.}$$

$$W(x) > W(y) \Rightarrow W(x') > W(y')$$

as we wished to prove.

Reciprocally,

$$W(x') > W(y') \Rightarrow W(x) > W(y),$$

which shows that the VNM axioms lead to dictatorships of the present. ■

2. The ranking $W(\alpha) = \liminf_{t \in R} \alpha_t$ – is a ‘dictatorship of the future’, and is ruled out by our axioms.

Now we know what is 'ruled out'

But what remains?

Can we provide a complete characterization of all the functionals that satisfy our axioms?

Representation Theorem

Theorem (Chichilnisky 1992, 1996)

- No prior ranking criteria satisfies our axioms.
- Yet there exist rankings $\Psi : L_\infty \rightarrow R$ satisfying our axioms.
- All such rankings are convex combinations of two terms: dictatorships of the present and dictatorships of the future.
- The two terms are (1) countably additive measures and (2) purely finitely additive measures on R .

For example, if states are discrete, indexed by the integers Z , there exists μ , $0 < \mu < 1$, such that $\Psi : l_\infty \rightarrow R$ is:

$$\Psi(\alpha) = \mu \sum_{t=1}^{\infty} \lambda^{-t} \alpha_t + (1 - \mu)(\Phi(\alpha_t)).$$

where Φ denotes a 'purely finite measure' on Z .

Equivalence with the Axiom of Choice

Examples: Rules of Thumb

1. Maximize expected utility, while minimizing total losses in a catastrophe
2. Network Optimization (Con Edison) Maximize average electricity throughput, while minimizing the chance of a 'black out'

Interpretation of Ψ

The first part is an integral operator with a countably additive kernel $\{\lambda^{-s}\}_{s \in Z}$ and emphasizing the weight of the present.

The second part is an extension of the function $\Phi(f) = \lim_{x \rightarrow \infty} f(x)$

(which is defined only on functions that have limits, a closed subspace of L_∞).

We extend Φ to all of L_∞ using Hahn Banach - or free ultrafilters. Both are equivalent to the Axiom of Choice.

Heavy Tails

The second purely finitely additive part Φ assigns positive weight to the future.

Because both parts are present, Ψ is sensitive to the present and the future.

The optimization of functionals such as Ψ is not amenable to standard tools of calculus of variations. This must be redeveloped in new directions. Some result already exist, see Chichilnisky (1996) and Heal (1996).

They give rise to nonautonomous dynamical systems, which are asymptotically autonomous (Hirsch and Benaim).

Who is the future?

Sustainability and Global Consciousness

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