COMPLEMENTARY COLORINGS

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ABSTRACT. Two 3-colorings of a cycle are *complementary* if whenever a vertex has its neighbors colored alike in one coloring, they are colored differently in the other coloring. Describing complementary colorings in terms of heawood colorings, we are able to count all such pairs. Complementary colorings can be defined for triangulations of manifolds. We construct complementary colorings for all oriented surfaces and for the 3-sphere. Finally, we apply these colorings to constructing triangulations whose odd part is a manifold.

1. 3-COLORING

A 3-coloring of a cycle C_n with n vertices is an assignment of the colors $1, 2, 3$ to the vertices of C_n such that any two adjacent vertices have distinct colors. A coloring f is *singular* at a vertex p if the colors of the two vertices adjacent to p are colored the same. If they are different, we say that f is *non-singular* at p. A permutation of the colors does not change the property of being non-singular or singular. A *heawood* coloring of C_n is an assignment of ± 1 to each of the edges of C_n. If orientations of C_n and C_3 are fixed, then a 3-coloring of C_n induces a heawood coloring by assigning to an edge the orientation of the corresponding edge of C_3. A heawood coloring is induced by a 3-coloring iff the sum of all the values is $0 \pmod 3$. We call 3-colorings f and g of the circle *complementary* if a vertex is singular under f iff it is non-singular under g. For example, the 3-coloring $(1, 2, 1, 3)$ of C_4 has non-singular singular pattern $(1, 0, 1, 0)$, where 1 is non-singular, and 0 is singular. A complementary coloring is given by $(1, 2, 3, 2)$, whose pattern is $(0, 1, 0, 1)$.

Since the number of the non-singular vertices has the same parity as the number of vertices, it follows that the number of vertices is even if there is a complementary coloring. Here is a construction for complementary colorings. Let h_1 and h_2 be two heawood colorings of C_n that are induced by 3-colorings. Define two heawood colorings of C_{2n} as follows: G consists of alternating terms of h_1 and h_2; H consists of alternating terms of h_1 and $-h_2$. G and H are heawood colorings that sum to 0 (mod 3), and so are induced by 3-colorings. These 3-colorings are complementary. For instance, to get the above example, we take $h_1 = (1, -1) = h_2$.

Theorem 1. *Every complementary coloring of C_{2m} is induced by a pair of heawood colorings of C_m.*

Proof. Write the heawood coloring corresponding to one of the 3-colorings as

$$(h_1, g_1, h_2, g_2, \dots, h_m, g_m)$$

where $n = 2m$. We know that

$$\sum h_i + \sum g_i = 0 \pmod 3$$

Switching to the complementary coloring, we easily see that we have the following heawood coloring: $(h_1, -g_1, h_2, -g_2, \dots, h_m, -g_m)$. It follows that

$$\sum h_i - \sum g_i = 0 \pmod 3$$

and so each sum is 0 (mod 3). Thus, g and h are the two desired heawood colorings. ☐

Corollary 1. *The number of complementary 3-colorings of C_{2m} is*

$$2\left(\frac{2^{m-1} + (-1)^m}{3}\right)^2$$

Proof. We count colorings as equivalent under permutations of colors. The number of 3-colorings of C_m is $(2^{m-1} + (-1)^m)/3$, and each one determines two heawood colorings. Fixing the sign of one edge, we get the result. ☐

For instance, C_6 has exactly one pair of complementary colorings - one is all non-singular and one is all singular.

2. 4-Coloring

In this section we extend our results to 4-colorings of surfaces. For basic facts about 4-coloring surfaces, see [Fis77, Fis80]. We say that two 4-colorings f and g of a surface are complementary if for each edge e, e is non-singular under f iff e is singular under g. If p is a vertex of S, then a pair of complementary colorings restricted to the link of p gives a pair of complementary 3-colorings of the link. Consequently, we see that if a surface has a pair of complementary colorings, then all vertices have even degree. This is not sufficient. For instance, it is easy to check that the octahedron has no complementary colorings.

The join $S^0 * C_n$ is a cycle with n vertices, each joined to two vertices p, q. Some joins can have complementary colorings.

Lemma 1. *$S^0 * C_n$ has a complementary coloring iff $n = 6k$.*

Proof. It is easy to check that if $n = 6k$, then the heawood coloring that assigns 1 to all triangles meeting p, and -1 to all triangles meeting q is a complementary coloring.

If w is a vertex of the cycle C_n in the join, then w has degree 4, and three are only two choices for a complementary coloring in the star of w. Extending these choices around the cycle, we find that the triangles in the neighborhood of p are labeled $a, b, a, b, \dots$, where a, b are ± 1. If this coloring is complementary, then the alternating sum of all the triangles meeting p must be $0 \pmod 3$. This implies that n is a multiple of 6. □

There are some local restrictions on the degrees of a triangulation with complementary colorings. For example,

Lemma 2. *If a triangulation of a surface has a triangle with degrees $4, 6, 6$, then there are no complementary colorings.*

Proof. If there is a coloring with a complement, then one of them is all non-singular at one of the vertices of degree 6. It follows that the edges of the other vertex of degree 6 must be non-singular. Now the vertex of degree 4 has three non-singular edges, but this is impossible. □

We now investigate several ways that we can get new triangulations with complementary colorings from triangulations with complementary colorings.

Given a complementary coloring of a triangulation M of a surface, we can construct more triangulations with complementary colorings by picking some edge e, and subdividing it six times. This means that if $e = ab$ is in triangles abc and abd, then we first remove these two triangles, and replace with triangles

$$\begin{array}{ccccccc} ap_1c & p_1p_2c & p_2p_3c & p_3p_4c & p_4p_5c & p_5p_6c & p_6bc \\ ap_1d & p_1p_2d & p_2p_3d & p_3p_4d & p_4p_5d & p_5p_6d & p_6bd \end{array}$$

Given the coloring on the triangles ap_1c and ap_1d, the complementary coloring is uniquely determined, and is easily seen to be consistent.

Another way to construct new complementary colorings from an old one is to take any branched covering of the surface, where the branch points lie in the vertex set of the graph. We get a new complementary coloring by lifting the old coloring.

A *heawood coloring* of a triangulation of a surface is an assignment of ± 1 to the triangles such that the sum of the values of the triangles containing a vertex is $0 \pmod 3$, for every vertex of the triangulation. If f is a 4-coloring, then $ns(f)$ is the set of all edges e such that the two triangles containing e use all four colors. If h is a heawood coloring, $ns(h)$ is the set of all edges e such that the two triangles containing e are labeled with 1 and -1. A *local coloring* is a set S of edges such that for every vertex v there is a 4-coloring f_v of the triangles containing v such that the set of all edges in S that contain v is precisely $ns(f_v)$. Two local colorings S, T are complementary iff $S \cap T = \emptyset$.

If we have two surfaces M, N with complementary colorings, then we construct a new surface with complementary colorings by choosing any triangle from M, and any from N, removing them, and joining the resulting surfaces along the boundary of the triangles. We call this a connected sum of N and M. The heawood coloring of the triangles is just the one from the old triangulation, chosen appropriately. We have

Lemma 3. *The connected sum of two triangulations N, M has a complementary heawood coloring if N and M both have complementary colorings.*

Proof. Suppose N has complementary heawood colorings g_1, g_2, and M has h_1, h_2. Suppose that the triangles chosen from N and M are labeled 1 by g_1 and h_1 respectively. We define complementary heawood colorings f_1, f_2 by setting $f_1 \mid N = g_1$, $f_1 \mid M = h_2$, and $f_2 \mid N = g_2$, $f_2 \mid M = h_1$. We only need to check that f_1 and f_2 are complementary at the three vertices of the chosen triangle, and this follows from the definition and the construction. ☐

We can find complementary colorings on any orientable surface.

Theorem 2. *Every orientable surface has a triangulation with a complementary coloring.*

Proof. There are infinitely many six-regular triangulations of the torus for which the all non-singular and the all singular local colorings are realized by 4-colorings. The all non-singular coloring and the all singular colorings are complementary. Since every orientable surface is a branched covering surface of the torus, we construct complementary colorings on any orientable surface by taking coverings of a suitably large triangulation of the torus. ☐

There are many more such triangulations, for we can apply subdivision and branched coverings to get quite complicated triangulations.

Question . *Is there a triangulation of some surface that has more than one pair of complementary colorings?*

3. Higher Dimensions

We do not know if there are complementary colorings on triangulations of n-manifolds, for n bigger than 3. There are many on the 3-sphere, and we construct them using joins and connected sums.

We first determine when we can get colorings and complementary colorings from joins. Recall that if M is a triangulation of the m-sphere, and N is a triangulation of the n-sphere, then the join $M * N$ is a triangulation of the (n+m+1)-sphere. The top simplices of the join are all joins of a top simplex of N and a top simplex of M.

A heawood coloring of a triangulation of a n-manifold is an assignment of ±1 to the top simplices so that the sum of the values of all the top simplices containing a fixed codimension 2 simplex is 0 (mod 3). In case that the manifold is simply connected and orientable, the assignment is induced by a $(n+2)$-coloring of the vertices.

If h is a heawood coloring of N and g is a heawood coloring of M, then we can define a labeling of the top simplices of $N * M$ by setting the value on a top simplex of the join to the product of the value of g on the top simplex of N and the value of h on the top simplex of M. We call this the product labeling determined by g and h.

If the n-simplices of an n-manifold can be colored with two colors such that the two n-simplices containing an $(n-1)$-simplex are always differently colored, then we say that the heawood coloring h is all singular, where h is 1 on simplices with one color, and -1 on simplices of the other. This is the case if the manifold has an $(n+1)$-coloring.

Lemma 4. *The product labeling is a heawood coloring iff one of the heawood colorings is all singular.*

Proof. Let g be a heawood coloring of M, h one of N, and let f be the product labeling. We must check that the sum of the labeling over every codimension-2 simplices of the join is 0 (mod 3). There are three types of codimension-2 simplices.

Case 1: A codimension-2 simplex γ is of the form $\sigma^m\tau^{n-2}$, where σ is a simplex of M, and τ is one of N. We compute:

$$\begin{aligned}\sum_{\gamma\subset\delta} f(\delta) &= \sum_{\tau\subset\alpha} g(\sigma)h(\alpha)\\ &= g(\sigma)\sum_{\tau\subset\alpha} h(\alpha)\\ &= 0\end{aligned}$$

where the last equality follows from the fact that h is a heawood coloring.

Case 2: γ is of the form $\sigma^{m-2}\tau^n$. This is just like Case 1.
Case 3: $\gamma = \sigma^{m-1}\tau^{n-1}$. We compute

$$\begin{aligned}\sum_{\gamma\subset\delta} f(\delta) &= \sum_{\tau\subset\alpha,\sigma\subset\beta} g(\beta)h(\alpha)\\ &= \sum_{\tau\subset\alpha} h(\alpha)\cdot\sum_{\sigma\subset\beta} g(\beta)\end{aligned}$$

In this case, we see that one of the two sums must be 0 for all choices of codimension-1 simplices. This is only possible if any two top simplices sharing a face have distinct values in one of the colorings. This is the case if and only if one of the colorings is all singular. □

Lemma 5. *If the all singular heawood coloring of a triangulation M of an n-manifold has a complementary heawood coloring, then n=1 or 2.*

Proof. The complementary coloring is the same on all simplices. Assume that $n \geq 3$. Taking the link of a codimension-3 simplex, we see that the two colorings induce colorings on a two sphere. From the even coloring we see that all vertices have even degree. From its complement, we find all vertices have degree divisible by 3. Thus all vertices have degree divisible by 6, but this is not possible for a 2-sphere. It follows that n is 1 or 2. □

Theorem 3. *If two triangulations of spheres have complementary colorings, then the product labeling of the join has a complementary coloring iff both manifolds are circles.*

Proof. From the above, we see that for the product to be a coloring, one of them must be all singular. Its complement is all non-singular, so the other colorings complement must be all singular as well. Now the above implies that they are both circles, since no 2-sphere satisfies Lemma 5. □

Corollary 2. *There are complementary colorings on some triangulations of the 3-sphere.*

Lemma 3 holds in any dimension. We can construct many complementary colorings of S^3 by taking connected sums of copies of $C_{6k} * C_{6k}$. Taking branched coverings, we can get complementary colorings on many 3-manifolds.

Question . *Does every (orientable) 3-manifold have a triangulation with complementary colorings ?*

Question . *Are there complementary colorings on some triangulated n-manifold, for $n > 3$?*

4. Applications

We apply these results to determine when the odd part of a n-manifold can be an (n-1)-manifold. Recall [Fis77] that if M is a triangulation of an n-manifold, the odd part of M, written $Odd(M)$, is the codimension-2 subcomplex of M consisting of all codimension-2 simplices of M that lie in an odd number of top simplices of M. For example, if $n = 2$, then $Odd(M)$ is the set of all vertices of odd degree. If $n = 3$, $Odd(M)$ is the 1-complex consisting of all edges that lie in an odd number of tetrahedra. The basic existence result is [Fis77]

Theorem 4. *Given any link, there is a triangulation of the 3-sphere whose odd part realizes the link.*

For an arbitrary 3-manifold M, $Odd(M)$ is a graph with all even degrees. We say that a triangulation K of a closed surface is the odd part of M if K is isomorphic to a subcomplex of M, and the edges of K are precisely the edges in $Odd(M)$. Similarly, we can say that an (n-1)-manifold is the odd part of a n-manifold.

Theorem 5. *If a n-manifold N has odd part that is a (n-1)-manifold K such that $N - K$ has two components, then K has complementary local colorings.*

Proof. We consider one component of $N - K$, and assign to each codimension-2 simplex of K the parity of the number of n-simplices lying in that component and containing the simplex. This is the usual construction of the local coloring induced by an even triangulation. The local coloring associated with the other side is clearly complementary. ☐

Theorem 6. *If an n-manifold M contains an (n-1)-manifold N without boundary such that $M - N$ has two components, and if N has complementary colorings, then there is a triangulation of M such that N is the odd part of M.*

Proof. Let the complementary colorings be f_1, f_2. We can find even triangulations M_1, M_2 of the closure of the two components of $M - N$ such that M_i induces f_i on N. We conclude that

$$Odd(M) = Odd(M_1 \cup M_2) = ns(f_1) + ns(f_2) = N$$

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Corollary 3. *For every orientable surface, there is a triangulation of the surface that is realizable as the odd part of a triangulation of the 3-sphere.*

Proof. This follows from Theorem 2 and Theorem 6. ☐

References

[Fis77] Steve Fisk, *Variations on coloring, surfaces, and higher dimensional manifolds*, Adv. in Math. **25** (1977), 226–266.

[Fis80] Steve Fisk, *Cobordism and functoriality of colorings*, Adv. in Math. **27** (1980), 177–211.