A NOTE ON WEYL'S INEQUALITY

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We present a simple inequality for the eigenvalues of Hermitian matrices that implies Weyl's inequality and the monotonicity theorem. The underlying idea, to intersect suitably chosen subspaces to obtain eigenvalue inequalities, is not new. See [TF71, IIM87]

Lemma 1. *If* $S_1, \ldots, S_k$ *are subspaces of an* n*-dimensional vector space* V*, and if* $dim(S_1) + \cdots + dim(S_k) > n(k-1)$*, then the intersection of all the* S_i*'s is non-zero.*

Proof. Consider the map from the direct sum of the subspaces S_i to the sum of $k-1$ copies of V which sends $(v_1, ..., v_k)$ to $(v_1 - v_2, v_2 - v_3, ..., v_k - v_{k-1})$. The intersection of all the S_i's is the kernel of this map, and has dimension at least $dim(S_1) + \cdots + dim(S_k) - (k-1)n$. □

If H is an n by n Hermitian matrix, we denote its ordered eigenvalues by $\lambda_1(H) \leq \cdots \leq \lambda_n(H)$. An n by n matrix X is *negative semidefinite* if $v^*Xv \leq 0$ for every vector v. For instance, the zero matrix is negative semidefinite.

Theorem 1. *Suppose* $H_1, \ldots, H_k$ *are* n *by* n *Hermitian matrices such that* $H_1 + H_2 + \cdots + H_k$ *is negative semidefinite. Then*

$$\lambda_{i_1}(H_1) + \lambda_{i_2}(H_2) + \cdots + \lambda_{i_k}(H_k) \leq 0$$

for all $i_1, \ldots, i_k \in \{1, \ldots, n\}$ *such that* $i_1 + \cdots + i_k < n + k$

Proof. Let S_j be the subspace spanned by the eigenvectors of H_j corresponding to the eigenvalues $\lambda_{i_j}(H_j), \lambda_{i_j+1}(H_j), \ldots, \lambda_n(H_j)$. Since

$$\sum_{j=1}^{k} dim(S_j) = \sum_{j=1}^{k} (n - i_j + 1) = nk + k - (i_1 + \cdots + i_k) > n(k-1),$$

Lemma 1 ensures that there is a unit vector x in the intersection of all the S_j's. Now $\lambda_{i_j}(H_j)$ is the smallest eigenvalue of H_j restricted to S_j, and therefore

$$\lambda_{i_j}(H_j) \leq x^* H_j x, \quad \text{for } j = 1, \ldots, k$$

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since each S_j is invariant under H_j. Adding these inequalities gives

$$\sum_{i=1}^{k} \lambda_{i_j}(H_j) \le x^* \left(\sum_{j=1}^{k} H_j \right) x \le 0$$

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Corollary 1 (Weyl's Inequality). *If A, B are n by n Hermitian, then*

$$\lambda_j(A) + \lambda_k(B) \le \lambda_{j+k+1}(A+B)$$

Proof. Take $H_1 = A$, $H_2 = B$, $H_3 = -(A+B)$, $i_1 = j$, $i_2 = k$, $i_3 = n - j - k$. The result follows from Theorem 1 and the fact that $\lambda_s(-C) = -\lambda_{n+1-s}(C)$ for any Hermitian matrix C.

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Corollary 2 (Monotonicity Theorem). *If A, B are n by n Hermitian, and B is positive semidefinite, then $\lambda_i(A) \le \lambda_i(A+B)$ for all $i = 1, \ldots, n$.*

Proof. Take $H_1 = A, H_2 = -A - B, i_1 = i, i_2 = n - i + 1$. ☐

The choice of eigenvalues in Theorem 1 is the best possible.

Theorem 2. *If $i_1 + \cdots + i_k \ge n + k$, then there are n by n Hermitian matrices $H_1, \ldots, H_k$ such that $H_1 + \cdots + H_k$ is negative semidefinite, and*

$$\lambda_{i_1}(H_1) + \cdots + \lambda_{i_k}(H_k) > 0$$

Proof. We may assume that $i_1 + \cdots + i_k = n + k$. Let H_s be the diagonal matrix whose diagonal is all 1's, except for the entries in rows $(i_1 + \cdots + i_{s-1}) + 2 - s, \ldots, (i_1 + \cdots + i_s) - s$ where the entries are $1 - k$. $H_1 + \cdots + H_k = 0$ is negative semidefinite, and $\lambda_{i_s}(H_s) = 1$, so $\lambda_{i_1}(H_1) + \cdots + \lambda_{i_k}(H_k) = k$. ☐

References

[IIM87] Yasuhiko Ikebe, Toshiyuki Inagaki, and Sadaaki Miyamoto, *The monotonicity theorem, Cauchy's interlace theorem, and the Courant-Fisher theorem*, American Mathematical Monthly **94** (1987), 352–354.

[TF71] R. C. Thompson and L. J. Freede, *On the eigenvalues of sums of Hermitian matrices*, Linear Algebra and Appl. **4** (1971), 369–376.

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