HOMOLOGY OF PROJECTIVE SPACE OVER FINITE FIELDS

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The q-analog of a simplex is a projective space over a finite field. We show that the homology of n-dimensional projective space over a finite field is the same as that of a simplex when n is even; when n is odd there is extra homology in the middle dimension. This difference between even and odd dimensions is not surprising since the simplex and finite projective space have the same Euler characteristic in even dimensions, but differ greatly in odd dimensions.

Our approach is to use some calculations of the rank of certain incidence matrices [FY90]. These calculations depend in turn on the representation theory of the general linear group over a finite field, in particular on the Submodule Theorem [Jam84].

We begin with some definitions. $\mathrm{PG}_n(q)$ is n-dimensional projective space over the finite field with q elements, and $V^n(q)$ is the n-dimensional vector space over the finite field with q elements. The number of r-subspaces of $V^n(q)$ is $\begin{bmatrix} n \\ r \end{bmatrix}$, and the number of r-dimensional flats of $\mathrm{PG}_n(q)$ is $\begin{bmatrix} n+1 \\ r+1 \end{bmatrix}$, where the Gaussian binomial coefficients are defined by

$$\begin{aligned}
\begin{bmatrix} n \\ r \end{bmatrix} &= \frac{[n]!}{[r]![n-r]!} \\
[n]! &= [n][n-1]\dots[1] \\
[n] &= (q^n-1)/(q-1) = q^{n-1}+q^{n-2}+\cdots+1
\end{aligned}$$

We need two facts about these coefficients; the first is elementary, and the second one is due to Gauss:

$$\begin{aligned}
\gcd(q+1,[n]) &= \begin{cases} 1 & \text{if } n \text{ is odd} \\ q+1 & \text{if } n \text{ is even} \end{cases} \\
\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} &= \begin{cases} 0 & \text{if } n \text{ is odd} \\ (1-q)(1-q^3)(1-q^5)...(1-q^{n-1}) & \text{if } n \text{ is even} \end{cases}
\end{aligned}$$

We fix a field $\mathcal{F}$ whose characteristic divides $q+1$. The chain group C_k is all sums of k flats in $\mathrm{PG}_n(q)$ with coefficients in $\mathcal{F}$. The boundary map $\partial : C_k \longrightarrow C_{k-1}$ sends a k-flat to the sum of all the $k-1$ flats contained in it. It is easy to check that if F is a k-flat, then

$$\partial^2(F) = (q+1)\sum_{G\subset F} G$$

where the sum is over all $k-2$ flats G contained in F. Since $char(\mathcal{F})$ divides $q+1$, ∂ is a boundary operator. Define the augmentation map $\epsilon : C_0 \longrightarrow \mathcal{F}$ by $\epsilon(\sum a_i v_i) = \sum a_i$. $H_i(\mathrm{PG}_n(q))$ is the i-th homology group of the chain complex

$$C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \xrightarrow{\epsilon} \mathcal{F}. \tag{1}$$

There are a few previous results ([ML83],[Mor83]) about these homology groups:

- $H_0(\mathrm{PG}_n(q)) = 0$ for $n > 1$. Define the coboundary operator δ applied to a k-flat to be the sum of the $k+1$ flats containing it. For n even, the result follows from the calculation $\partial\delta(v-w) = v-w$ for any two points v, w; a simple modification works for n odd and greater than 1.
- $H_i(\mathrm{PG}_n(q)) = H_{n-i-1}(\mathrm{PG}_n(q))$. The Poincare duality map sends a flat to its orthogonal complement. The homological symmetry follows as usual.
- $H_i(\mathrm{PG}_{2n}(q)) = 0$. The coboundary operator δ is a chain homotopy that establishes the triviality of the homology.

Our result will be a consequence of this result of Frumkin and Yakir ([FY90]):

Frumkin and Yakir. *Assume that non-negative integers l, k satisfy $l \le k$ and $l + k \le n$. $A^{l,k}$ is the $\begin{bmatrix} n \\ l \end{bmatrix}$ by $\begin{bmatrix} n \\ k \end{bmatrix}$ incidence matrix whose rows correspond to the l-subspaces of $V^n(q)$, and whose columns correspond to the k-subspaces. Define*

$$Y = \left\{ i \,|\, 0 \le i \le l\,,\ \begin{bmatrix} k-i \\ l-i \end{bmatrix} \ne 0 \ in\ \mathcal{F} \right\}.$$

Then

$$rank_{\mathcal{F}}(A^{l,k}) = \sum_{i\in Y} \left(\begin{bmatrix} n \\ i \end{bmatrix} - \begin{bmatrix} n \\ i-1 \end{bmatrix} \right)$$

We only need a special case of this theorem:

Corollary. *If $2k+1 \le n$, then the rank of $A^{k,k+1}$ is*

$$\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n \\ k-1 \end{bmatrix} + \begin{bmatrix} n \\ k-2 \end{bmatrix} - \begin{bmatrix} n \\ k-3 \end{bmatrix} + \cdots$$

Proof. Since $k + (k+1) \leq n$, the conditions of the theorem are satisfied. We compute the set Y:

$$\begin{aligned} Y &= \left\{ i \,|\, 0 \leq i \leq l \,,\, \begin{bmatrix} k+1-i \\ k-i \end{bmatrix} \neq 0 \right\} \\ &= \{ i \,|\, 0 \leq i \leq l \,,\, [k+1-i] \neq 0 \} \end{aligned}$$

Since $char(\mathcal{F})$ divides $q+1$, $[k+1-i]$ is non-zero iff k and i have the same parity. Consequently, $Y = \{k, k-2, k-4, \ldots\}$, and the result follows from the theorem. ■

Theorem.

$$H_i(\mathrm{PG}_{2m}(q)) = 0$$

$$rank\, H_i(\mathrm{PG}_{2m+1}(q)) = \begin{cases} (q-1)(q^3-1)\ldots(q^{2m+1}-1) & i = m \\ 0 & else \end{cases}$$

Remarks.

- The Euler characteristic of the chain complex (1) is
$$\sum_{i=0}^{n+1} (-1)^i \begin{bmatrix} n+1 \\ i+1 \end{bmatrix}$$
which is 0 if n is even, and $(1-q)(1-q^3)\ldots(1-q^n)$ if n is odd.
- When $q = 1$, the middle rank is 0.

Proof. If suffices to show that $H_k(\mathrm{PG}_n(q)) = 0$ for $k \leq (n-1)/2$, or equivalently $2k+3 \leq n+1$. If so, then by duality ([Mor83]) we have $H_{n-k-1}(\mathrm{PG}_n(q)) = H_k(\mathrm{PG}_n(q)) = 0$. If n is even, this shows that all the homology groups are zero. If $n = 2m+1$, then this establishes that all homology groups are zero, except for $H_m(\mathrm{PG}_{2m+1}(q))$. Since we know the Euler characteristic, the result follows.

We now show that these initial homology groups are zero. Consider the map $\partial : C_{k+1} \longrightarrow C_k$. Since the k flats of $\mathrm{PG}_n(q)$ are $k+1$ subspaces of $V^n(q)$, the matrix of ∂ as a linear transformation is $A^{k+1,k+2}$. We thus compute

$$\begin{aligned} rank(H_k(\mathrm{PG}_n(q))) &= dim(Ker\ \partial : C_k \longrightarrow C_{k-1}) - dim(Im\ \partial : C_{k+1} \longrightarrow C_k) \\ &= dim(C_k) - dim(Im\ \partial : C_k \longrightarrow C_{k-1}) \\ &\quad - dim(Im\ \partial : C_{k+1} \longrightarrow C_k) \\ &= \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} - rank(A^{k,k+1}) - rank(A^{k+1,k+2}) \end{aligned}$$

Since $(k+1)+(k+2) \leq n+1$, we can apply the Corollary.

$$\begin{aligned}
&= \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} - \left(\begin{bmatrix} n+1 \\ k \end{bmatrix} - \begin{bmatrix} n+1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n+1 \\ k-2 \end{bmatrix} - \dots \right) \\
&\qquad - \left(\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} - \begin{bmatrix} n+1 \\ k \end{bmatrix} + \begin{bmatrix} n+1 \\ k-1 \end{bmatrix} - \dots \right) \\
&= 0
\end{aligned}$$

■

Remarks.

- It is easy to show that the even dimensional homology is 0; it would be nice to find a simple proof for the odd dimensions, excluding the middle.
- We conjecture that the middle homology is generated by the chains associated to hyperbolic quadrics in $\mathrm{PG}_{2m+1}(q)$. The Kernel Intersection Theorem [Jam84] supports this conjecture.
- The homology theory for projective space is the q-analog of $\mathbb{Z}/2\mathbb{Z}$ homology. What is the q-analog of homology over the integers?

References

[FY90] Avital Frumkin and Arieh Yakir, *Rank of inclusion matrices and modular representation theory*, Israel Journal of Mathematics **71** (1990), no. 3, 309–320.

[Jam84] G.D. James, *Representations of general linear groups*, London Mathematical Society Lecture Notes, vol. 94, Cambridge University Press, 1984.

[ML83] W. Mielants and H. Leemans, *Z_2-cohomology of projective spaces of odd order*, Annals of Discrete Mathematics **18** (1983), 635–652.

[Mor83] Hisasi Morikawa, *On a certain homology of finite projective spaces*, Nagoya Math. J. **90** (1983), 57–62.

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