Continuous Symmetry

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From Euclid to Einsten

A modern approach to undergraduate geometry.

[The aim of education] is the acquisition of the art of utilization of knowledge. A merely well-informed man is the most useless bore on God's earth.
— Alfred North Whitehead, **The Aim of Education**, 1912

Draft Textbook.

Roger Howe and William Barker, ***Continuous Symmetry***, Xeroxed manuscript.

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Prerequisites.

Multivariable Calculus or its equivalent.

Description of Continuous Symmetry.

These two courses and the accompanying manuscript are intended to take Felix Klein's ***Erlangen Program*** seriously. Specify a *base set* (such as the plane or space) and specify a group of transformations on this set, the *symmetries* of the set. A property of a figure in the base set is *invariant* if it does not change under the application of a symmetry transformation to the figure. ***Geometry*** *is then the study of symmetry invariant properties of figures.* Change the designated group of symmetries and you obtain a new geometry on the base set. All of the common geometric systems are obtained in this unified manner.

The first half of the *Continuous Symmetry* manuscript constitutes a stand-alone course on transformational geometry in the plane and in space — the background and ability of the students will dictate the exact amount of material that can be covered in one semester. A unique feature of our treatment is that it is *synthetic*, i.e., coordinate-free. The geometry is thus not lost in algebraic manipulation. On the other hand, group theory is introduced in the second chapter and is exploited mercilessly for the remainder of the book. No prior knowledge of group theory is assumed of the readers — all the relevant concepts are developed in the text.

The course starts with a careful development of Euclidean geometry in the plane based on **metric axioms**. This is followed by an extensive study of the **isometries** (distance preserving mappings) and **similarities** (proportion preserving mappings) of the plane. Each of these two collections of transformations forms a **group**, the central structure of the course.

We then reverse the development described in the previous paragraph: we show that the group of isometries can be used to *define* Euclidean geometry (**congruence geometry**) in the plane. From this viewpoint, the isometries are considered the **symmetries** of the plane — congruence geometry becomes the study of properties of figures that are **invariant** under the symmetry group of isometries. We also consider the similarities as a group of symmetries of yet another type of Euclidean geometry: **similarity geometry**. In this system the similarities are viewed as the symmetries of the plane — similarity geometry becomes the study of properties of figures that are invariant under the symmetry group of similarities.

Congruence geometry and similarity geometry thus illustrate the general process for defining *any* geometry: the study of invariance under a specified group of transformations. As pointed out earlier, this **transformational** view of geometry is the central idea of Felix Klein's *Erlangen Program*. We illustrate the value of this approach by developing transformational proofs for several interesting and non-trivial theorems in plane geometry.

We then move from two to three dimensions, developing congruence geometry and similarity geometry in **space**. Armed with these tools, we study **symmetric figures** in both two and three dimensions. This entails the analysis of discrete groups of symmetries, including all the **ornamental groups** in the plane and the **crystallographic groups** in space.

The next chapter studies area and volume (**measure**) and **dimension**, especially how these concepts interplay with change of **scaling** via similarities. This leads naturally to an introduction to **fractional dimension** and **fractals**.

The material up to this point constitutes a self-contained and logically satisfying mathematical theory that does not depend for its coherence on continuing with the remainder of the text. (Moreover, for a more leisurely paced one-semester course topics such as the geometry of space can be omitted.) However, the ideas developed thus far have compelling extensions and applications that form the heart of the second portion of the manuscript and a second semester course.

Our development of geometry through the chapter on measurement is **coordinate-free**: there is no reliance on **coordinates** (apart from simple coordinates on a line) nor any use of the techniques of **analytic geometry** in the plane or in space. However, to extend our results in the desired directions it is necessary to introduce two- and three-dimensional coordinate systems. This, in turn, requires the use of **matrices** in the analysis of isometries and similarities, leading directly to **matrix algebra** and, more generally, **linear algebra**.

In this way the Euclidean symmetry groups become **matrix groups** and lead to a sophisticated method for defining symmetry groups for more general geometric systems. We use this approach to develop both **affine** and **projective** geometry in the plane, and then the **non-Euclidean** geometries (**hyperbolic** and **elliptic**) as subgeometries of projective geometry.

Finally, based on our transformational approach to geometries in two and three dimensions, we give a rigorous development and comparison for the four-dimensional geometries of Newtonian spacetime and relativistic spacetime. The primary unifying objects are — no surprise — the symmetry groups for Newtonian and relativistic spacetime: the Galilean group and the Poincaré group respectively. The two spacetime models are carefully derived from basic physical assumptions, making clear the differences between the two theories and the geometric reasons for the surprising properties of the relativistic model.

Contents (as of January 2005).

Design of the course sequence at Bowdoin College
Fall 2003/Spring 2004

Components of the Courses.

You will be expected to **read the text materials**. In particular, the designated sections of the text should be read *prior* to the class sessions for which they are assigned.

Individual assignments will be given each week. Discussion of the problems with other class members is actively encouraged, but your submissions must be your own written work, not copies of a classmate's papers.

Many of the weekly assignments will contain a more challenging **collaborative** portion, often built around a multi-step project.

For each semester there will be a **midterm examination** as well as a **final examination** at the end of the semester.

Individual and Collaborative Assignments.

Weekly assignments will be due at the start of the **Tuesday** class session. Collaborative projects will be due at the start of the **Thursday** session.

Submitted work should be clear and complete. In particular, your logic and solution techniques should be explained in complete sentences.

The Assignment Groups.

The collaborative assignments will be done within designated *Assignment Groups*. Each group will consist of three or four members, each with an assigned role. The roles should be rotated so that everyone samples each task. The roles are as follows:

Manager. The manager **schedules** the group meetings and **notifies** all group members of the times and places.

Scribe. The scribe **writes up** the solutions. The more complete and creative the solutions, the higher the grade. The scribe should attempt to do much of the write-up during the group sessions themselves. All group members should examine the final paper before submission. On longer assignments it is acceptable for two group members to share the role of Scribe.

Clarifier. During meetings the clarifier **monitors group dynamics**, making sure that no member of the group is left out or hopelessly lost. The clarifier is to be a **careful observer**; it is not assumed that the clarifier has greater understanding of the material than anyone else in the group.

Reporter. The Reporter **records** the particulars of group work sessions in an assignment report. *The report should be the first page of the assignment submission.* It must (1) say when the group met, (2) list who was present (and for how long) at each meeting, (3) specify what roles everyone had, and (4) contain the signatures of all group members, vouching for their full participation in the assignment solutions.

In groups of three students, the roles of Reporter and Manager should be merged.

Advice on Collaborative Learning.

Be prepared. Prior to meeting do the readings and think about the problems.

Contribute to the assignment solutions. Everyone has this responsibility.

Listen carefully and with respect to each other.

Ask for help when you need it.

Give help when it is requested.

Criticize ideas, not people. Be tolerant, respectful, and caring.

Never agree to something you don't understand in a group submission.

Additional References.

Edwin Moise, *Elementary Geometry from an Advanced Standpoint,* Third Edition, Addison Wesley, 1990.

This is a beautiful, rigorous, and complete development of Euclidean geometry in the plane and in space. It will serve as our primary reference for basic geometric facts throughout the course.

Brannan, Esplen, and Gray, *Geometry*, Cambridge Un. Press, 1999.

This is a complete and lavishly illustrated development of affine, projective, and non-Euclidean geometries. It has the same philosophy of geometry as *Continuous Symmetry* and complements the latter portion of the course where the *Continuous Symmetry* manuscript is still incomplete.

Class Attendance.

You cannot be an effective and involved member of the class unless you are present.

Course Grade.

The course grade each semester will be determined by the assignments and the exams as follows:

Assignments & Projects	50%
Midterm Exam	20%
Final Exam	30%

Honesty.

The course instructor supports and adheres to the principles of Bowdoin's Honor Code. In particular, the students will be assumed trustworthy in all their dealings with the instructor and fellow classmates. However, should a violation of this trust be discovered, disciplinary action will be taken. In particular, all serious violations of the Honor Code will be automatically reported to the Judiciary Board. The goal is not vengeance against those who violate the Code but fairness for those who adhere to it.