Appendix to “Exchange Rate Regimes and Wage Comovements in a Dynamic Ricardian Model”

Yoshinori Kurokawa,* Jiaren Pang† and Yao Tang‡

February 20, 2015
(Supplementary Appendix)

This appendix contains details about the proofs of the three propositions in Kurokawa, Pang and Tang (2015) and the related lemmas. For convenience in reference, we briefly restate the model and key conditions here. For country $j = H, F$, the period preference for the representative consumer is

$$U_j^t = \left(\frac{C_j^t}{1 - \zeta}\right)^{1 - \zeta} - \kappa \left(\frac{L_j^t}{1 + \gamma}\right)^{1 + \gamma} + \chi h \left(\frac{M_j^t}{P_j^t}\right),$$

where

$$C_j^t = \left[\left(\int_0^1 C_j^i (i) \frac{u - 1}{\eta} di\right)^{\frac{1}{\eta - 1}} \cdot C_j^i (z)^{1 - \epsilon}\right],$$

$\zeta, \kappa, \gamma, \chi$ and $\eta > 0$ and $0 < \epsilon < 1$. The period budget constraint is

$$\int_0^1 P_j (i) C_j (i) di + P_j (z) C_j (z) + M_j^t + \sum_s q_j^t (s, z) B_j^s (s, z) = W_j^t L_j + M_j^{t-1} + B_j^t + \Pi_j^t + T_j^t.$$

There is also a borrowing constraint

$$B_j^t (s, t+1) \geq -P_j^t \tilde{b}.$$
Given the prices $P_{jt}(i)$ and $P_{jt}(z)$, the minimization of the cost of $C_{jt}^j$ yields the following
unit cost of $C_{jt}^j$, which we refer to as the price of $C_{jt}^j$

$$P_{jt}^j = \left[ \epsilon^{-\epsilon} (1 - \epsilon)^{-1} \right] \left( \int_0^1 (P_{jt}(i))^{1-\eta} dt \right)^{1-\eta} (P_{jt}(z))^{1-\epsilon} .$$  \(A.1\)

Hence, the budget constraint can be written as

$$P_{jt}^j C_{jt}^j + M_{jt}^j + \sum_s q^j (s_{t+1}) B^j (s_{t+1}) = W_{jt}^j L_{jt}^j + M_{t-1}^j + B_{jt}^j + \Pi_{jt}^j + T_{jt}^j .$$  \(A.2\)

The production technology is

$$Y_{jt}^j (i) = A_{jt}^j (i) L_{jt}^j (i) ,$$  \(A.3\)

$$Y_{jt}^j (z) = A_{jt}^j (z) L_{jt}^j (z) ,$$  \(A.4\)

where $A_{jt}^j (i)$ and $A_{jt}^j (z)$ are the stochastic productivities.

Under perfect competition, the domestic price of good $i$ posted by the firms in
country $j$ in the local currency is

$$P_{jt}^i (i) = W_{jt}^i / A_{jt}^i (i) ,$$

but the prevailing market prices that consumers in the home and foreign countries actually
pay are now given by

$$P_{Hi} (i) = \min \{ P_{jt}^H (i) , (1 + \tau) P_{ft}^F (i) e_t \} ,$$  \(A.5\)

$$P_{Fi} (i) = \min \{ (1 + \tau) P_{jt}^H (i) / e_t , P_{ft}^F (i) \} ,$$  \(A.6\)

where $e_t$ is the nominal exchange rate, defined as the price of foreign currency in the home
currency.

The market for nontradable goods is also perfectly competitive. Consequently, the
local-currency prices for the nontradable goods are

$$P_{jt}^i (z) = W_{jt}^i / A_{jt}^i (z) .$$  \(A.7\)
The market clearing conditions are

\begin{align}
L^H_t &= \int_0^{k^H} L^H_t(i) \, di + L^H_t(z), \quad (A.8) \\
L^F_t &= \int_{k^H}^1 L^F_t(i) \, di + L^F_t(z), \quad (A.9) \\
Y_t^H(i) &= C_t^H(i) + C_t^F(i) \,(1 + \tau) \, \forall i < k^H_t, \quad (A.10) \\
Y_t^F(i) &= C_t^H(i) \,(1 + \tau) + C_t^F(i) \, \forall i > k^F_t, \quad (A.11) \\
Y^j_t(i) &= C^j_t(i) \, \forall k^H_t \leq i \leq k^F_t, \quad (A.12) \\
Y^j_t(z) &= C^j_t(z). \quad (A.13)
\end{align}

The balanced-trade condition is

\begin{align}
\int_0^{k^H} C^F_t(i) \,(1 + \tau) \, P^H_t(i) \, di &= \int_{k^F}^1 C^H_t(i) \,(1 + \tau) \, P^F_t(i) \, \epsilon_t \, di. \quad (A.14)
\end{align}

For the household’s maximization problem, the first order conditions with respect to \( C^j_t(i) \), \( C^j_t(z) \), \( L^j_t \), \( M^j_t \), and \( B^j(s_{t+1}) \) are

\begin{align}
\left( C^j_t \right)^{1-\zeta} \frac{\epsilon \left( C^j_t(i) \right)^{\frac{1}{\eta}}} {\int_0^{1} \left( C^j_t(i) \right)^{\frac{1}{\eta}} \, di} &= P^j_t(i) \lambda^j_t, \quad (A.15) \\
\left( C^j_t \right)^{1-\zeta} \frac{1 - \epsilon \left( C^j_t(z) \right)} {C^j_t(z)} &= P^j_t(z) \lambda^j_t, \quad (A.16) \\
\kappa \left( L^j_t \right)^{\gamma} &= W^j_t \lambda^j_t, \quad (A.17) \\
\frac{\chi^h \left( M^j_t / P^j_t \right)} {P^j_t} &= \lambda^j_t - \beta E_t \lambda^j_{t+1}, \quad (A.18) \\
\beta E_t \lambda^j_{t+1} (s_{t+1}) &= q^j(s_{t+1}) \lambda^j_t, \quad (A.19)
\end{align}

where \( \lambda^j_t \) is the Lagrange multiplier for the budget constraint and \( \beta \) is the time discount factor.

An equivalent approach is to maximize the household’s utility subject to the budget constraint (A.2). The first order condition with respect to the aggregate consumption \( C_t \)
\( (C^j_t)^\zeta = P^j_t \lambda^j_t, \quad (A.20) \)

while other first order conditions are identical.

As in Chari, Kehoe and McGrattan (2002), the equilibrium exchange rate is determined by

\[ e_t = \frac{P^H_t (C^H_t)^\zeta}{P^F_t (C^F_t)^\zeta} \delta = \frac{\lambda^F_t}{\lambda^H_t} \delta, \quad (A.21) \]

where the quantity \( \delta \) is a constant depending on the state of the economies in the initial period; the marginal utility in the home country relative to that in the foreign country in the initial period.

### A.1 Proofs of Proposition 1 and related lemmas

First, we prove two lemmas useful for the proof of Proposition 1.

**Lemma 1.** The real exchange rate is

\[ e_t \frac{P^F_t}{P^H_t} = \left( \frac{e_t W^F_t}{W^H_t} \right)^{1-\epsilon} \left( \frac{A^H_t(z)}{A^F_t(z)} \right)^{1-\epsilon} D_t, \]

where

\[
D_t = \left[ \int_0^{k^H_t} ((1 + \tau) P^H_t (i))^{1-\eta} di + \int_{k^H_t}^{k^F_t} (P^F_t (i) e_t)^{1-\eta} di + \int_{k^F_t}^1 ((1 + \tau) P^F_t (i) e_t)^{1-\eta} di \right]^{\frac{1}{1-\eta \epsilon}}.
\]

Proof: Substituting equation (A.1) into the definition of real exchange rate, we have

\[ e_t \frac{P^F_t}{P^H_t} = e_t \left( \int_0^1 (P^F_t (i))^{1-\eta} di \right)^{\frac{1}{1-\eta \epsilon}} \left( \int_0^1 (P^H_t (i))^{1-\eta} di \right)^{\frac{1}{1-\eta \epsilon}} \frac{1}{1-\eta \epsilon}. \]
Therefore, by using equations (A.5), (A.6), and (A.7), we can rewrite the expression for the real exchange rate as

\[
\begin{align*}
\frac{P^F_{t}}{P^H_{t}} &= \frac{\epsilon_t}{W^H_t W^F_t} \left[ \int_0^{k^H_t} \left( \frac{(1+\tau) P^H_t (i)}{\epsilon_t} \right)^{1-\eta} di + \int_{k^H_t}^{k^F_t} (P^F_t (i))^{1-\eta} di + \int_{k^F_t}^{1} (P^F_t (i))^{1-\eta} di \right]^{1-\gamma} \\
&= \epsilon_t^{1-\gamma} \left( \frac{W^F_t}{W^H_t} \right)^{1-\gamma} \left( \frac{A^H_t (z)}{A^F_t (z)} \right)^{1-\gamma} \\
&= \epsilon_t^{1-\gamma} \frac{\epsilon_t}{W^H_t W^F_t} \left[ \int_0^{k^H_t} \left( \frac{(1+\tau) P^H_t (i)}{\epsilon_t} \right)^{1-\eta} di + \int_{k^H_t}^{k^F_t} (P^F_t (i))^{1-\eta} di + \int_{k^F_t}^{1} \left( (1+\tau) P^F_t (i) \epsilon_t \right)^{1-\eta} di \right]^{1-\gamma} \\
&= \epsilon_t^{1-\gamma} \frac{\epsilon_t}{W^H_t W^F_t} \left[ \int_0^{k^H_t} \left( \frac{(1+\tau) P^H_t (i)}{\epsilon_t} \right)^{1-\eta} di + \int_{k^H_t}^{k^F_t} (P^F_t (i))^{1-\eta} di + \int_{k^F_t}^{1} \left( (1+\tau) P^F_t (i) \epsilon_t \right)^{1-\eta} di \right]^{1-\gamma}.
\end{align*}
\]

(A.22)

Defining

\[
D_t = \left[ \int_0^{k^H_t} \left( \frac{(1+\tau) P^H_t (i)}{\epsilon_t} \right)^{1-\eta} di + \int_{k^H_t}^{k^F_t} (P^F_t (i))^{1-\eta} di + \int_{k^F_t}^{1} \left( (1+\tau) P^F_t (i) \epsilon_t \right)^{1-\eta} di \right]^{1-\gamma},
\]

(A.23)

we can rewrite equation (A.22) as

\[
\frac{P^F_{t}}{P^H_{t}} = \epsilon_t^{1-\gamma} \frac{\epsilon_t}{W^H_t W^F_t} \left( \frac{A^H_t (z)}{A^F_t (z)} \right)^{1-\gamma} D_t.
\]

(A.24)

Therefore, the real exchange rate is determined by the relative wage \( \epsilon_t W^F_t \), productivities in nontradable goods, and the term \( D_t \) that can be viewed as the ratio of the price index for tradable goods in the foreign country to that in the home country. The presence of the term \( D_t \) is due to the trade costs \( \tau \). When \( \tau \) is zero, \( D_t = 1 \). Note that the relationship between \( D_t \) and \( \tau \) can be complex, depending on the distribution of tradable productivities in the two countries. For instance, for positive values of \( \tau \), if distributions of tradable productivities in the two countries are mirror images to each other (i.e., \( A^H_t (i) = A^F_t (1-i) \) for all \( i \)), then \( D_t \) is also 1.

**Lemma 2.** Labor and consumption are linked by the equation \( L^i_t = \kappa^{\frac{i}{\gamma+\gamma}} \left( C^i_t \right)^{\frac{i}{\gamma+\gamma}} \).

Proof: Note that equations (A.4), (A.7), (A.13), and (A.17) imply

\[
C^i_t (z) = L^i_t (z) A^i_t (z),
\]

\[
P^i_t (z) = \frac{W^i_t}{A^i_t (z)},
\]

\[
\chi^i_t = \frac{\kappa}{W^i_t}.
\]
Using these equations to eliminate $C^j_t (z)$, $P^j_t (z)$, and $\lambda^j_t$ from equation (A.16), we have

$$
(1 - \epsilon) \left( C^j_t \right)^{1 - \zeta} = L^j_t (z) \left( L^j_t \right)^{\gamma} \kappa.
$$

(A.25)

Rewrite (A.15) as

$$
(1 - \epsilon) \left( C^j_t \right)^{1 - \zeta} \frac{\epsilon \left( C^j_t (i) \right)^{\frac{q-1}{\eta}}}{\int_{0}^{1} \left( C^j_t (i) \right)^{\frac{q-1}{\eta}} di} = C^j_t (i) P_{jt} (i) \lambda^j_t.
$$

Integrating both sides with respect to $i$, for the home country we have

$$
\int_0^{k^F} C_H^i (i) P_{Ht} (i) di + \int_{k^F}^{1} C_H^i (i) P_{Ht} (i) di = \frac{\epsilon}{\lambda^H_t} \left( C_H^i \right)^{1 - \zeta}.
$$

(A.26)

Substituting the balanced trade condition into the above equation and using the fact that $P_{Ht} (i) = P_H^H (i) \forall i \leq k^F$, we have

$$
\int_0^{k^F} C_H^i (i) P_H^i (i) di + \int_{k^F}^{1} C_H^i (i) P_H^i (i) (1 + \tau) di = \frac{\epsilon}{\lambda^H_t} \left( C_H^i \right)^{1 - \zeta}.
$$

(A.27)

Note that equations (A.3) and (A.10) imply

$$
C_H^i (i) + C^F_t (i) (1 + \tau) = L_H^i (i) A_H^i (i) \quad \forall i < k^H,
$$

$$
C_H^i (i) = L_H^i (i) A_H^i (i) \quad \forall k^H \leq i \leq k^F,
$$

$$
P^H_t (i) = \frac{W^H_t}{A^H_t (i)} \quad \forall i \leq k^F.
$$

Substituting these equations into equation (A.26), we obtain

$$
\int_0^{k^F} L_H^i (i) di = \frac{\epsilon \left( C_H^i \right)^{1 - \zeta}}{W^H_t \lambda^H_t}.
$$

(A.28)

Similarly, combining equations (A.4), (A.7), (A.13), (A.16) and (A.17), we can obtain

$$
L_H^i (z) = \frac{1 - \epsilon}{W^H_t \lambda^H_t} \left( C_H^i \right)^{1 - \zeta}.
$$

Together, equations (A.27) and (A.28) imply that

$$
\frac{\int_0^{k^F} L_H^i (i) di}{L_H^i (z)} = \frac{\epsilon}{1 - \epsilon}.
$$
Using the last equation to eliminate \( \int_0^k F_t L_t^H (i) \, di \) from equation (A.8), we have

\[
L_t^H (z) = (1 - \epsilon) L_t^H.
\]

Substituting the last line into equation (A.25), we can obtain

\[
L_t^H = \kappa^{\frac{1}{1+\gamma}} (C_t^H)^{\frac{1-\epsilon}{1+\gamma}}.
\]  \hspace{1cm} (A.29)

Similarly, we can obtain

\[
L_t^F = \kappa^{\frac{1}{1+\gamma}} (C_t^F)^{\frac{1-\epsilon}{1+\gamma}}.
\]

\[\blacksquare\]

**Proposition 1.** The relationship between growth in home nominal wages and the foreign counterpart is

\[
\frac{W_t^H}{W_{t-1}^H} = \frac{W_t^F}{W_{t-1}^F} e_t \left( \frac{A_t^H (z) A_{t-1}^F (z)}{A_t^H (z) A_{t-1}^F (z)} \right)^{\gamma (1+\epsilon)(1-\zeta)} \left( \frac{D_t}{D_{t-1}} \right)^{1+\gamma (1+\epsilon)(1+\zeta) (1+\gamma \epsilon) (1-\zeta)}.
\]  \hspace{1cm} (A.30)

Proof: By Lemma 2, we can express the real exchange rate as a function of the nominal exchange rate, nominal wages, and productivities:

\[
e_t \frac{P_t^F}{P_t^H} = \left( \frac{e_t W_t^F}{W_t^H} \right)^{1-\epsilon} \left( \frac{A_t^H (z)}{A_t^F (z)} \right)^{1-\epsilon} D_t.
\]  \hspace{1cm} (A.31)

The equilibrium exchange rate, stated in equation (A.21), links the real exchange rate to relative consumption

\[
e_t \frac{P_t^F}{P_t^H} = \delta \left( \frac{C_t^H}{C_t^F} \right)^{\zeta}.
\]

Note that Lemma 1 shows that consumption is linked to labor supply via the labor market clearing conditions, and that labor supply is linked to nominal wages through the households’ optimization problem. Substituting equation (A.29) and equation (A.17) into the last equation, we can obtain the relationship between the real exchange rate, nominal wages, and nominal exchange rate:

\[
e_t \frac{P_t^F}{P_t^H} = \left( \frac{W_t^H}{W_t^F} \right)^{\zeta (1+\gamma)} \delta^{1+\zeta (1+\gamma)}.
\]  \hspace{1cm} (A.32)

We can view equation (A.31) as the relationship between the real exchange rate and relative wage implied by technology, and equation (A.32) as the relationship between the real exchange rate and relative wage implied by preferences. Combining equations (A.31)
and (A.32) to eliminate the real exchange rate, we can relate the changes in nominal wages to changes in technology as in

\[
\frac{W_t^H}{W_{t-1}^H} = \frac{W_t^F}{W_{t-1}^F} e_t \left( \frac{A_t^H(z) A_{t-1}^F(z)}{A_t^H(z) A_{t-1}^F(z)} \right)^{\gamma(1-\epsilon)(1-\zeta)/\gamma(1-\epsilon)(1+\gamma)} \left( \frac{D_t}{D_{t-1}} \right)^{\gamma(1-\epsilon)/\gamma(1-\epsilon)(1+\gamma)}.
\]

### A.2 Proofs of Proposition 2 and related lemmas

In order to derive Proposition 2 regarding nominal wage comovements, we first solve for the marginal utilities of nominal wealth, \( \lambda_t^H \) and \( \lambda_t^F \), under assumptions (a) and (c1).

**Lemma 3.** Under assumptions (a) and (c1), the marginal utility of nominal wealth \( \lambda_t^j \) is

\[
\lambda_t^j = \frac{\chi \psi^j}{M_t^j}
\]

where \( \psi^j \) is a constant.

**Proof:** From equation (A.18), we have

\[
\lambda_t^j = \frac{\chi}{M_t^j} + \beta E_t \lambda_{t+1}^j
\]

\[
= \frac{\chi}{M_t^j} + \beta E_t \left( \frac{\chi}{M_{t+1}^j} \right) + \beta^2 E_t \left( \frac{\chi}{M_{t+2}^j} \right) + \beta^3 E_t \left( \frac{\chi}{M_{t+3}^j} \right) + \cdots
\]

\[
= \frac{\chi}{M_t^j} \left[ 1 + \frac{\beta}{1+g^j} E_t \left( \frac{1}{\exp(\mu_{t+1}^j)} \right) + \left( \frac{\beta}{1+g^j} \right)^2 E_t \left( \frac{1}{\exp(\mu_{t+2}^j) \exp(\mu_{t+3}^j)} \right) \right]
\]

\[
+ \left( \frac{\beta}{1+g^j} \right)^3 E_t \left( \frac{1}{\exp(\mu_{t+1}^j) \exp(\mu_{t+2}^j) \exp(\mu_{t+3}^j)} \right) + \cdots
\]

\[
= \frac{\chi}{M_t^j} \left[ 1 + \frac{\beta}{1+g^j} \int \frac{1}{\exp(\mu^j)} d\Phi(\mu^j) + \left( \frac{\beta}{1+g^j} \right)^2 \int \int \frac{1}{\exp(\mu^j)} d\Phi(\mu^j) \frac{1}{\exp(\mu^j)} d\Phi(\mu^j) \right]
\]

\[
+ \left( \frac{\beta}{1+g^j} \right)^3 \int \int \int \frac{1}{\exp(\mu^j)} d\Phi(\mu^j) \frac{1}{\exp(\mu^j)} d\Phi(\mu^j) \frac{1}{\exp(\mu^j)} d\Phi(\mu^j) + \cdots
\]

The terms in the brackets are equal to a constant. Defining the constant as \( \psi^j \), we have

\[
\lambda_t^j = \frac{\chi \psi^j}{M_t^j}.
\]

(A.33)
Combined with equation (A.21), an immediate corollary of Lemma 3 is that

\[ \epsilon_t = \frac{M_t^H \delta \psi_t^F}{M_t^F \psi_t^H}, \quad (A.34) \]

Substituting equation (A.33) into the foreign version of equation (A.17), we have

\begin{align*}
W_t^F &= \frac{\kappa M_t^F}{\chi \psi_t^F} (L_t^F)^\gamma \\
&= \frac{\kappa M_t^F}{\chi \psi_t^F} \kappa_t \gamma \gamma \left( C_t^F \right)^{\frac{\gamma(1-\xi)}{\gamma+\gamma}} \\
&= \frac{\kappa_t \gamma \gamma}{\chi \psi_t^F} M_{t-1}^F (1 + g_t^F) \exp \left( \mu_t^F \right) \left( C_t^F \right)^{\frac{\gamma(1-\xi)}{\gamma+\gamma}}, \quad (A.35)
\end{align*}

where the second equality follows from Lemma 1. Note that the foreign version of equation (A.20) and Lemma 3 imply that

\[ C_t^F = C_{t-1}^F \left( \frac{P_{t-1}^F \chi_{t-1}^F}{P_t^F \chi_t^F} \right)^{\frac{1}{\gamma}} \]

\[ = C_{t-1}^F \left( \frac{P_{t-1}^F (1 + g_t^F) \exp \left( \mu_t^F \right)}{P_t^F} \right)^{\frac{1}{\gamma}}. \]

Combining the last line with equations (A.17), (A.30), (A.33), (A.34), and (A.35) and the definition for \( P_t^F \), we can write the changes in wages as explicit functions of state variables and shocks

\[ \frac{W_t^H}{W_{t-1}^H} = \frac{M_t^H}{M_{t-1}^H} \times \]

\[ \left\{ \exp \left[ (\eta - 1) \left( a_t^H + \rho^H a_{t-1}^H + u_t^H \right) \right] \int_0^{h_t^F} \left( A^H (i) \right)^{\eta-1} di + \right. \]

\[ \left. \left( \frac{W_{t-1}^H}{\epsilon_{t-1}} \right)^{-(1-\eta)} \exp \left[ - \frac{\gamma (1-\epsilon) (1-\zeta) (1-\eta)}{\gamma (1-\epsilon) + \zeta (1+\epsilon \gamma)} \left( A_t^H (z) - A_{t-1}^H (z) + A_t^F (z) - A_{t-1}^F (z) \right) \right] \times \right. \]

\[ \left. \exp \left[ (\eta - 1) \left( a_t^F + \rho^F a_{t-1}^F + u_t^F \right) \right] \int_0^{k_t^F} \left( A^F (i) \right)^{\eta-1} di \right\}^{\frac{\epsilon_t (1-\zeta)(1-\eta)}{\gamma \theta (1+\epsilon \gamma)}} \times \]

\[ \left\{ A^H (z) \exp \left( a_t^H (z) + \rho^H (z) a_{t-1}^H (z) + v_t^H \right) \right\}^{\frac{\gamma(1-\epsilon)(1-\zeta)}{\gamma+\epsilon \gamma}} \times \]

\[ \kappa_t \gamma \gamma \left( M_{t-1}^H \right)^{\frac{\gamma(1-\xi)}{\gamma+\gamma}} \left( C_{t-1}^H \right)^{\frac{\gamma(1-\xi)}{\gamma+\gamma}} \left( P_{t-1}^H \right)^{\frac{\gamma(1-\xi)}{\gamma+\gamma}} \left[ \epsilon^t \left( 1 - \epsilon \right)^{\epsilon-1} \right]^{\frac{\gamma \zeta (1-\xi)}{\gamma+\epsilon \gamma}}, \quad (A.36) \]
Proposition 2. Under assumptions (a), (b) and (d), nominal wage comovements between the countries are more positive or less negative under the fixed exchange rate regime (assumption (c2)), compared to the flexible exchange rate regime (assumption (c1)). To be specific,

\[
\text{corr}_{FX}^{FL} \left[ \ln \left( \frac{W^H}{W_{t-1}^H} \right), \ln \left( \frac{W^F}{W_{t-1}^F} \right) \right] - \text{corr}_{FL}^{FL} \left[ \ln \left( \frac{W^H}{W_{t-1}^H} \right), \ln \left( \frac{W^F}{W_{t-1}^F} \right) \right] \geq 0,
\]

where FX and FL denote the fixed and flexible exchange rate regimes, respectively. The strict equality holds only when monetary shocks \( \mu_t^H \) and \( \mu_t^F \) are perfectly correlated.
Proof: We maintain the normalization $W_{t-1} = 1$. Substituting equation (A.37) back into (A.30), we obtain an alternative expression for the home wage growth

$$
\frac{W^H_{t}}{W^F_{t-1}} = \exp \left[ \frac{\gamma (1-\epsilon) \left(1 - \zeta \right)}{\gamma (1-\epsilon) + \zeta + \zeta \gamma} \left( A^H_t (z) - A^H_{t-1} (z) - A^F_t (z) + A^F_{t-1} (z) \right) \right] \frac{D_t}{D_{t-1}} \times
$$

$$
\left( 1 + \eta \right) \exp \left( \frac{D_t}{D_{t-1}} \right) \times
$$

$$
\left\{ \frac{W^H_{t-1}}{\epsilon^H_{t-1}} \right\}^{1-\eta} \exp \left[ \frac{\gamma (1-\epsilon) \left(1 - \zeta \right)}{\gamma (1-\epsilon) + \zeta + \zeta \gamma} \left( A^H_t (z) - A^H_{t-1} (z) - A^F_t (z) + A^F_{t-1} (z) \right) \right] \frac{D_t}{D_{t-1}} \times
$$

$$
\exp \left[ \left( \eta - 1 \right) \left( a^H_t + \rho^H a^H_{t-1} + u^H_t \right) \right] \int_0^{k^H_t} \left( \frac{A^H_t (z)}{1+\tau} \right)^{\eta-1} \, dt +
$$

$$
\exp \left[ \left( \eta - 1 \right) \left( a^F_t + \rho^F a^F_{t-1} + u^F_t \right) \right] \int_0^{k^F_t} \left( A^F_t (z) \right)^{\eta-1} \, dt \right\} \frac{\gamma (1-\epsilon) \left(1 - \zeta \right)}{\gamma (1-\epsilon) + \zeta + \zeta \gamma} \times
$$

$$
\left[ A^F_t \exp \left\{ \left( \eta - 1 \right) \left( a^F_t + \rho^F a^F_{t-1} + u^F_t \right) \right\} \right] \frac{\gamma (1-\epsilon) \left(1 - \zeta \right)}{\gamma (1-\epsilon) + \zeta + \zeta \gamma} \times
$$

$$
\left( \frac{\gamma (1-\epsilon) \left(1 - \zeta \right)}{\gamma (1-\epsilon) + \zeta + \zeta \gamma} \right)^{\eta - 1} \frac{\gamma (1-\epsilon) \left(1 - \zeta \right)}{\gamma (1-\epsilon) + \zeta + \zeta \gamma}
$$

$$
(A.40)
$$

To reduce notation clusters, define

$$
\exp \left[ d_t \left( v^H_t, v^F_t \right) \right] = \left. \frac{\gamma (1-\epsilon) \left(1 - \zeta \right)}{\gamma (1-\epsilon) + \zeta + \zeta \gamma} \left( A^H_t (z) - A^H_{t-1} (z) - A^F_t (z) + A^F_{t-1} (z) \right) \right]
$$

$$
\exp \left[ f_t \left( u^H_t, u^F_t, v^H_t, v^F_t \right) \right] = \left. \frac{\gamma (1-\epsilon) \left(1 - \zeta \right)}{\gamma (1-\epsilon) + \zeta + \zeta \gamma} \left( A^H_t (z) - A^H_{t-1} (z) - A^F_t (z) + A^F_{t-1} (z) \right) \right]
$$

such that the random variables $d_t$ and $f_t$ are functions of the underlying productivity shocks. Note that $D_t$ is a function of productivity shocks and the trade costs, but not monetary shocks. To see this, we substitute the zero-profit conditions into (A.23) to obtain

$$
D_t = \frac{f_t^k \left( 1+\tau \right) \frac{W^H_t}{A^H_t (z)} \frac{1-\eta}{\left( \frac{A^H_t (z)}{1+\tau} \right)^{\eta-1}} \, dt + f_t^k \left( \frac{W^H_t}{A^H_t (z)} e_t \right) \frac{1-\eta}{\left( \frac{A^H_t (z)}{1+\tau} \right)^{\eta-1}} \, dt + f_t^k \left( \frac{W^F_t}{A^F_t (z)} e_t \right) \frac{1-\eta}{\left( \frac{A^F_t (z)}{1+\tau} \right)^{\eta-1}} \, dt}{f_t^k \left( \frac{W^H_t}{A^H_t (z)} \right)^{1-\eta} \, dt + f_t^k \left( \frac{W^H_t}{A^H_t (z)} e_t \right)^{1-\eta} \, dt + f_t^k \left( \frac{W^F_t}{A^F_t (z)} e_t \right)^{1-\eta} \, dt}
$$

We then use equation (A.34) and equation (A.35) to eliminate $W^H_t$ from the last line and cancel out $W^F_t e_t$ from both the numerator and denominator to get

$$
D_t = \frac{f_t^k \left( 1+\tau \right) \frac{L^H_t}{L^F_t} \frac{1}{\left( \frac{L^H_t}{L^F_t} \right)^{\gamma-1}} \, dt + f_t^k \left( \frac{1}{A^H_t (z)} \right)^{1-\eta} \, dt + f_t^k \left( \frac{1}{A^F_t (z)} \right)^{1-\eta} \, dt}{f_t^k \left( \frac{1}{A^H_t (z)} \right)^{1-\eta} \, dt + f_t^k \left( \frac{1}{A^F_t (z)} \right)^{1-\eta} \, dt}
$$

$$
\frac{1}{\left( \frac{L^H_t}{L^F_t} \right)^{\gamma-1}}
$$

11
Because in our model, relative labor supplies \( (L_t^H/L_t^F) \) are only affected by real shocks, the term \( D_t \) is a function of trade costs and real shocks but not monetary shocks.

With the definitions, under the fixed exchange rate regime, equation (A.40) can be rewritten as

\[
\ln \left( \frac{W_t^H}{W_{t-1}^H} \right) = \ln (1 + g_t^F) + \mu_t^F + d_t + f_t.
\]

Note that under the flexible exchange rate regime

\[
e_t = \frac{(1 + g_t^H) e_t^H}{(1 + g_t^F) e_t^F}.
\]

Hence, under the flexible exchange rate regime, equation (A.40) becomes

\[
\ln \left( \frac{W_t^H}{W_{t-1}^H} \right) = \ln (1 + g_t^H) + \mu_t^H + d_t + f_t.
\]

Similarly, equation (A.37) can be rewritten as

\[
\ln \left( \frac{W_t^F}{W_{t-1}^F} \right) = \ln (1 + g_t^F) + \mu_t^F + f_t.
\]

Therefore, under the fixed exchange rate regime, the correlation coefficient between home and foreign wage growth is

\[
corr^{FX} \left[ \ln \left( \frac{W_t^H}{W_{t-1}^H} \right), \ln \left( \frac{W_t^F}{W_{t-1}^F} \right) \right] = \frac{\text{cov} (\mu_t^F + d_t + f_t, \mu_t^F + f_t)}{\sqrt{\text{var} (\mu_t^F + d_t + f_t) \text{var} (\mu_t^F + f_t)}} = \frac{\text{cov} (\mu_t^F, \mu_t^F) + \text{cov} (d_t + f_t, f_t)}{\sqrt{\text{var} (\mu_t^F) + \text{var} (d_t + f_t) \text{var} (f_t) + \text{var} (f_t)}}
\]

where the second equality follows from the assumption that both monetary shocks are independent of the productivity shocks.

Under the flexible regime, the correlation coefficient between home and foreign wage growth is

\[
corr^{FL} \left[ \ln \left( \frac{W_t^H}{W_{t-1}^H} \right), \ln \left( \frac{W_t^F}{W_{t-1}^F} \right) \right] = \frac{\text{cov} (\mu_t^H + d_t + f_t, \mu_t^F + f_t)}{\sqrt{\text{var} (\mu_t^H + d_t + f_t) \text{var} (\mu_t^F + f_t)}} = \frac{\text{cov} (\mu_t^H, \mu_t^F) + \text{cov} (d_t + f_t, f_t)}{\sqrt{\text{var} (\mu_t^H) + \text{var} (d_t + f_t) \text{var} (f_t) + \text{var} (f_t)}}
\]
where the last equality follows from assumption (c1), which states that the monetary shocks have the same marginal distributions.

Because \( \text{var} (\mu_i^H - \mu_i^F) \geq 0 \) implies
\[
\text{var} (\mu_i^H) + \text{var} (\mu_i^F) \geq 2 \cdot \text{cov} (\mu_i^H, \mu_i^F),
\]
or
\[
\text{var} (\mu_i^F) \geq \text{cov} (\mu_i^H, \mu_i^F),
\]
it follows that
\[
\begin{align*}
\text{corr}^{FX} \left[ \ln \left( \frac{W_i^H}{W_{i-1}^H} \right), \ln \left( \frac{W_i^F}{W_{i-1}^F} \right) \right] - \text{corr}^{FL} \left[ \ln \left( \frac{W_i^H}{W_{i-1}^H} \right), \ln \left( \frac{W_i^F}{W_{i-1}^F} \right) \right] \\
= \frac{\text{var} (\mu_i^F) - \text{cov} (\mu_i^H, \mu_i^F)}{[\text{var} (\mu_i^F) + \text{var} (d_t + f_t)]^{1/2} \cdot [\text{var} (\mu_i^F) + \text{var} (f_t)]^{1/2}} \geq 0,
\end{align*}
\]
where the strict equality holds only when \( \mu_i^H \) and \( \mu_i^F \) are perfectly correlated. ■

### A.3 Proof of Proposition 3

**Proposition 3.** Suppose that nominal prices and wages are flexible. Thus, (1) the exchange rate regime does not affect real wage comovements, and (2) the exchange rate regime does not affect the pattern of trade, real exchange rate, real consumption comovements, labor comovements, or real output comovements.

**Proof:**

**Trade pattern.** In the model, the home country exports varieties from 0 to \( k_i^H \) and the foreign country exports varieties from \( k_i^F \) to 1. Using (A.38), the cutoff variety \( k_i^H \) is decided by
\[
\frac{A^H (k_i^H)}{A^F (k_i^H)} = \frac{W_i^H (1 + \tau) \exp (a_i^F + \rho F \alpha_i^F - 1 + u_i^F)}{W_i^F e_t \exp (a_i^H + \rho H \alpha_i^H - 1 + u_i^H)}. \tag{A.41}
\]
Using equation (A.30) to eliminate \( W_i^H / (W_i^F e_t) \) from the last line, it is obvious that the cutoff variety \( k_i^H \) depends only on past wages, past exchange rate, and shocks, and it does not depend on the exchange rate regime. A similar argument holds for \( k_i^F \). Therefore, the choice of exchange rate regime does not affect trade pattern. The result is important because it implies that in the discussion below, we do not need to worry that \( k_i \) varies with the exchange rate regime.

**Consumption.** Substituting equation (A.29) into equation (A.17) to eliminate \( L_i^j \), we have
\[
\kappa \left[ \kappa^{-1} \left( C_i^j \right)^{-1/\gamma} \right]^{1-\gamma} = W_i^j \lambda_i^j.
\]
Using equation (A.33) to replace $\lambda_j^t$ from the last line yields
\[
(C^j_t)^{\frac{\gamma(1-\zeta)}{1+\gamma}} = \frac{W^j_t}{M^j_t} \chi^H \kappa^H.
\]

Therefore
\[
\frac{\gamma(1-\zeta)}{1+\gamma} \ln \left( \frac{C^j_t}{C^j_{t-1}} \right) = \ln \left( \frac{W^j_t}{W^j_{t-1}} \right) - \ln \left( \frac{M^j_t}{M^j_{t-1}} \right). \tag{A.42}
\]

Under the fixed exchange rate regime, using the notation in the proof of Proposition 2, we have
\[
\frac{\gamma(1-\zeta)}{1+\gamma} \ln \left( \frac{C^H_t}{C^H_{t-1}} \right) = \ln \left( 1 + g^F + \mu^F_t + d_t + f_t - [\ln (1 + g^F) + \mu^F_t] \right)
= d_t + f_t. \tag{A.43}
\]

Under the flexible exchange rate regime, it holds that
\[
\frac{\gamma(1-\zeta)}{1+\gamma} \ln \left( \frac{C^H_t}{C^H_{t-1}} \right) = \ln \left( 1 + g^H + \mu^H_t + d_t + f_t - [\ln (1 + g^H) + \mu^H_t] \right)
= d_t + f_t. \tag{A.44}
\]

Similarly, we can obtain
\[
\frac{\gamma(1-\zeta)}{1+\gamma} \ln \left( \frac{C^F_t}{C^F_{t-1}} \right) = f_t.
\]

From the last three equations, it is obvious that real consumption comovements measured by
\[
\text{corr} \left[ \ln \left( \frac{C^H_t}{C^H_{t-1}} \right), \ln \left( \frac{C^F_t}{C^F_{t-1}} \right) \right]
\]
are the same under both the fixed and flexible exchange rate regimes. That is, the choice of exchange rate regime does not affect real consumption comovements.

**Labor.** Substituting equation (A.33) into equation (A.17) yields
\[
(L^H_t)^{\gamma} = \frac{W^H_t}{M^H_t} \chi^H \kappa^H.
\]

Hence
\[
\gamma \ln \left( \frac{L^H_t}{L^H_{t-1}} \right) = \ln \left( \frac{W^H_t}{W^H_{t-1}} \right) - \ln \left( \frac{M^H_t}{M^H_{t-1}} \right). \tag{A.45}
\]
Note that the right hand side of the last line is identical to equation (A.42). Following the same argument above, we know that the choice of exchange rate regime does not affect comovements of labor.

**Real wage.** From equation (A.20) we have

$$\lambda_t^H = \frac{(C_t^H)^{-\zeta}}{P_t^H}.$$ 

Substituting the last equation into equation (A.17), we obtain

$$\frac{W_t^H}{P_t^H} = \kappa (L_t^H)^\gamma (C_t^H)^\zeta.$$ 

Therefore, the change in real wages is

$$\ln \left( \frac{W_t^H / P_t^H}{W_{t-1}^H / P_{t-1}^H} \right) = \gamma \ln \left( \frac{L_t^H}{L_{t-1}^H} \right) + \zeta \ln \left( \frac{C_t^H}{C_{t-1}^H} \right)$$

$$= \ln \left( \frac{W_t^H}{W_{t-1}^H} \right) - \ln \left( \frac{M_t^H}{M_{t-1}^H} \right) + \zeta \frac{1 + \gamma}{\gamma (1 - \zeta)} \left[ \ln \left( \frac{W_t^H}{W_{t-1}^H} \right) - \ln \left( \frac{M_t^H}{M_{t-1}^H} \right) \right]$$

$$= \left[ 1 + \frac{\zeta (1 + \gamma)}{\gamma (1 - \zeta)} \right] \left[ \ln \left( \frac{W_t^H}{W_{t-1}^H} \right) - \ln \left( \frac{M_t^H}{M_{t-1}^H} \right) \right]$$

$$= \left[ 1 + \frac{\zeta (1 + \gamma)}{\gamma (1 - \zeta)} \right] (d_t + f_t), \quad (A.46)$$

where the second equality follows from equations (A.42) and (A.45), and the last equality follows from equation (A.43). Note that equations (A.43) and (A.44) imply that the last expression applies to both the fixed and flexible exchange rate regimes. Similarly, we have

$$\ln \left( \frac{W_t^F / P_t^F}{W_{t-1}^F / P_{t-1}^F} \right) = \left[ 1 + \frac{\zeta (1 + \gamma)}{\gamma (1 - \zeta)} \right] f_t. \quad (A.47)$$

From equations (A.46) and (A.47), we can see that the choice of exchange rate regime does not affect comovements of real wages.

**Real output.** The nominal output of the home country, $Y_t^H$, is

$$Y_t^H = \int_0^{k_t^F} P_t^H (i) Y_t^H (i) \, di + P_t^H (z) C_t^H (z)$$

$$= \int_0^{k_t^H} P_t^H (i) (C_t^H (i) + (1 + \tau) C_t^F (i)) \, di + \int_{k_t^H}^{k_t^F} P_t^H (i) C_t^H (i) \, di + P_t^H (z) C_t^H (z)$$

$$= \int_0^1 P_t^H (i) C_t^H (i) \, di + P_t^H (z) C_t^H (z)$$

$$= P_t^H C_t^H,$$
where the second last equality follows from the balanced trade condition (A.14) and the last equality follows from equation (A.1). Note that $P^H_t$ is the price index of the composite consumption good. Let the price index associated with the output be

$$[\alpha^{-\alpha} (1 - \alpha)^{-1}] \left( \int_0^{k^F_t} P^H_t (i)^{1-\eta} \, di \right)^\frac{1}{1-\eta} P^H_t (z)^{1-\alpha},$$

where $\alpha$ is the weight given to the tradable goods. A plausible choice is that $\alpha = \epsilon$, but our argument works for all $\alpha$ in $[0, 1]$. With this definition of the price index, the real output, $\tilde{Y}^H_t$, is

$$\tilde{Y}^H_t = \frac{P^H_t C^H_t}{[\alpha^{-\alpha} (1 - \alpha)^{-1}] \left( \int_0^{k^F_t} P^H_t (i)^{1-\eta} \, di \right)^\frac{1}{1-\eta} P^H_t (z)^{1-\alpha}}$$

$$= \frac{P^H_t C^H_t}{[\alpha^{-\alpha} (1 - \alpha)^{-1}] \left( \int_0^{k^F_t} \left( \frac{1}{A^H_t (i)} \right)^{1-\eta} \, di \right)^\frac{1}{1-\eta} \left( \frac{1}{A^H_t (z)} \right)^{1-\alpha}} W^H_t$$

$$= \frac{P^H_t C^H_t}{W^H_t A^H_t} \left[ \alpha^{-\alpha} (1 - \alpha)^{-1} \left( \int_0^{k^F_t} \left( \frac{1}{A^H_t (i)} \right)^{1-\eta} \, di \right)^\frac{1}{1-\eta} \left( \frac{1}{A^H_t (z)} \right)^{1-\alpha} \tilde{A}^H_t \right],$$

where $\tilde{A}^H_t = [\alpha^{-\alpha} (1 - \alpha)^{-1}] \left( \int_0^{k^F_t} \left( \frac{1}{A^H_t (i)} \right)^{1-\eta} \, di \right)^\frac{1}{1-\eta} \left( \frac{1}{A^H_t (z)} \right)^{1-\alpha}$. Therefore

$$\ln \left( \frac{\tilde{Y}^H_t}{Y^H_{t-1}} \right) = -\ln \left( \frac{W^H_t / P^H_t}{W^H_{t-1} / P^H_{t-1}} \right) + \ln \left( \frac{C^H_t}{C^H_{t-1}} \right) - \ln \left( \frac{\tilde{A}^H_t}{\tilde{A}^H_{t-1}} \right)$$

$$= \frac{1}{\gamma} (d_t + f_t) - \ln \left( \frac{\tilde{A}^H_t}{\tilde{A}^H_{t-1}} \right).$$

Note that the last line holds under both the fixed and flexible exchange rate regimes.

Similarly, we have

$$\ln \left( \frac{\tilde{Y}^F_t}{Y^F_{t-1}} \right) = \frac{1}{\gamma} (f_t) - \ln \left( \frac{\tilde{A}^F_t}{\tilde{A}^F_{t-1}} \right).$$

16
Therefore, the comovements of real output measured by
\[ \text{corr} \left[ \ln \left( \frac{Y^H_t}{Y^H_{t-1}} \right), \ln \left( \frac{Y^F_t}{Y^F_{t-1}} \right) \right] \]
are not affected by the choice of exchange rate regime.

**Real exchange rate.** Substituting equation (A.41) into equation (A.24) to eliminate \( W^H_t/W^F_t e_t \) yields
\[
e_t P^F_t = \left[ \frac{A^F (k^H_t) (1 + \tau) \exp \left( a^F_t + \rho^F \alpha^F_{t-1} + u^F_t \right)}{A^H (k^H_t) \exp \left( a^H_t + \rho^H \alpha^H_{t-1} + u^H_t \right)} \right]^{1-\epsilon} \left( \frac{A^H_t (z)}{A^F_t (z)} \right)^{1-\epsilon} D_t.
\]
Therefore the real exchange rate is only affected by productivities, not by the choice of exchange rate regime. ■

**References**
